ROBUST ESTIMATION OF RADIATION SOURCES BY DUAL METHOD OF MINIMUM SPATIAL EXTENT

Sergiy M. VOVK Valentine F. BORULKO
Oles Honchar Dnipropetrovsk National University, Dnipropetrovsk, Ukraine
E-mail: vovk_s_m@mail.ru, borulko@inbox.ru

Abstract: *Estimation of radiation-source parameters from near-field measurement data is considered. Case of distortion by additive and impulsive noises is investigated. Robust approach based on a concept of "minimal spatial extent" is suggested. The "spatial extent" is used two times, namely, in an original space (which is the space of solution or sources) and in a conjugate space (which is the space of equations discrepancy or measurements). The model of radiation sources is described by a set of ideal Hertzian dipoles. The case, when dipoles are located along a straight line, which is parallel to measurement line of electric field, and when their electric moments are perpendicular to the measurement plane, is considered. Algorithm, based on the conjugate gradient method, is proposed. Numerical simulations of direct and inverse problems are presented.*

Keywords: *electric dipole, spatial extent, robust estimation, regularization.*

1. Introduction

One of important problems of antenna theory is the estimation of spatial locations and amplitudes of radiation sources from near-field measurements [1-3]. This problem belongs to a class of inverse problems and is incorrect. In [4-5] we have proposed approach for solving this problem in scalar case, where we have assumed that the radiation sources are the point sources of homogeneous spherical waves. In [6] we have considered the vector case, assuming that the radiation sources are dipoles. There we have described the radiating system by a finite set of electric dipoles, which are located along a straight line and have the same orientation of dipole moments. We have also supposed that in near-field of radiating system the components of electric field can be measured with negligible influence of the probe. In this paper we assume that the measured data are distorted by additive Gaussian noise, as well as by random impulses appeared due to equipment failures.

Since the radiated field of dipole system is a superposition of individual fields of dipoles, it is natural to formulate the estimation problem as a problem of solving system of linear algebraic equations (SLAE). In general, the matrix of this SLAE is a rectangular matrix; its horizontal size is determined by a number of the "points of solution" (i.e. points, where the radiation sources can be located) and vertical size is determined by a number of the "points of measurement" (i.e. points, in which the measurements were performed). This kind of matrix leads to find a pseudosolution, e.g. the least squares solution with minimal Euclidean norm [7-8]. However, if SLAE is ill-conditioned, then such pseudosolution can not provide a good quality solution when the data are distorted by noise. In this case the commonly used method is the regularization [9]. Basic idea of regularization is to use a priori information about the solution properties and to involve it in a problem statement.

Traditionally, this a priori information is formulated as requirement of minimal energy of solution (e.g., the power of antenna ohmic losses) and/or minimal energy of solution derivatives. But this approach usually gives a very smooth solution when real solution is not indeed smooth. To avoid this drawback we use the natural requirement of a minimal spatial extent of solution and assume that the radiation sources are located sparsely in unknown points of a given discrete grid. In addition, due to random impulses in data we have a SLAE, in which some part of its equations may have big rough errors. To avoid this shortcoming we use the requirement of a minimal spatial extent of discrepancy of solution. In aggregate, these both requirements allow building the robust estimation technique [10], which is required to estimate the radiation sources under additive and impulsive noises.

2. Problem Statement

There are many ways to describe the radiation sources by a set of electric dipoles. Further we

consider an example, where one electric dipole corresponds to one radiation source. Also we use the concept of an ideal electric Hertzian dipole and study the case of a medium without absorption. Then the electromagnetic field, radiated by an ideal electric Hertzian dipole, is described by following expressions [11]:

$$\vec{E} = \frac{p}{4\pi\varepsilon}[\vec{r_0}\frac{2}{r^2}\left(\frac{1}{r}+ik\right)\cos\theta + \\ + \vec{\theta_0}\frac{1}{r}\left(\frac{1}{r^2}+\frac{ik}{r}-k^2\right)\sin\theta]e^{-ikr}, \quad (1)$$

$$\vec{H} = \vec{\varphi_0}\frac{p\omega}{4\pi}\left(\frac{i}{r^2}-\frac{k}{r}\right)\sin\theta e^{-ikr}, \quad (2)$$

where $\vec{r_0}$, $\vec{\theta_0}$ and $\vec{\varphi_0}$ are unit vectors of spherical coordinate system; unit vector $\vec{z_0}$ points to dipole orientation; $i=\sqrt{-1}$; $k=2\pi/\lambda$ is wavenumber; λ is free space wavelength; ε is permittivity; r is a distance to an observation point; ω is angular frequency; $\vec{p}=p\vec{z_0}$ is electric dipole moment; $\vec{E}$, $\vec{H}$ are complex amplitudes of electric (1) and magnetic (2) fields, respectively.

Let electric dipoles are located along a straight line (line of sources *S*) and have the same orientation of their dipole moments. Let a line of measurement *M* is parallel to *S* in some plane and electric moments of dipoles are perpendicular to this plane (Fig.1).

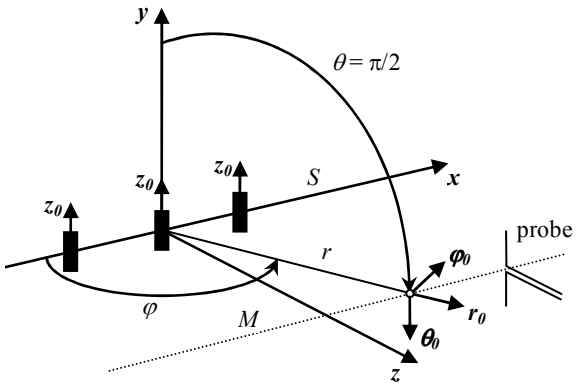

Fig. 1. Geometry of the problem.

Then the electric field of dipole has only component

$$E_\theta = \frac{p}{4\pi\varepsilon}\frac{1}{r}\left(\frac{1}{r^2}+\frac{ik}{r}-k^2\right)e^{-ikr}. \quad (3)$$

Let use additional system of Cartesian coordinates, for which *x*-axis is aligned with *S*, *z*-axis is perpendicular to *S* and to *M*, and *y*-axis is parallel to vector $\vec{z_0}$ of electric dipole moment. Then we have $E_y = -E_\theta$. Using (3), the following SLAE can be written

$$b_j = \sum_{n=1}^{N} u_n\left[k^2 - \frac{ik}{r_{nj}} - \frac{1}{r_{nj}^2}\right]\times\frac{e^{-ikr_{nj}}}{r_{nj}} + \eta_j; \\ r_{nj}^2 = x_{nj}^2 + z^2; \quad x_{nj} = x_n^{(s)} - x_j^{(m)}; \quad j = 1,...,J \quad (4)$$

where b_j are values of E_y at the "points of measurements"; u_n are unknown values of field at the "points of solution"; η_j are noise samples; *z* is a distance between *S* and *M*; $x_n^{(s)}$ and $x_j^{(m)}$ are *x*-coordinates of "points of solution" and "points of measurements," respectively; *N* is a number of "points of solution"; *J* is a number of "measurement points". Further we assume that dipoles are located at some "points of solution" and the number of dipoles is substantially smaller than *N*.

In matrix presentation, SLAE (4) has a form of $Au = b$, where *A* is a complex-valued matrix, which elements are formed from the spatial model for a given system of dipoles, *b* is a complex-valued column vector of known data obtained by measuring; *u* is an unknown complex-valued column vector of source amplitudes. The estimation problem of radiation sources is to solve the SLAE (4).

It is necessary to note that the direct solving: $u = A^{-1}b$ is possible only for the case of square matrix *A*, when the number of "points of solution" *N* is equal to the number of "measurement points" *J*. But such solution has a major drawback, because matrix A^{-1} is usually ill-conditioned, making the solution unstable. If the matrix *A* is a rectangular matrix, it is necessary to use the pseudosolution [7-8], i.e., the Least-Squares Solution $u = (A^H A)^{-1}A^H b$, obtained from minimization problem $\| Au - b \|^2 \to \min_u$, where A^H denotes the conjugate (Hermitian) transpose of the complex-

valued matrix A and $\|...\|^2$ denotes the squared Euclidean norm. But this solution has the same drawback. We point here that in both cases mentioned above the condition number of SLAE matrix increases as the distance between the points of measurements decreases. Moreover, the solution $u = (A^H A + \rho^2 I)^{-1} A^H b$, obtained by standard Tikhonov regularization technique: $\| Au - b \|^2 + \rho^2 \| u \|^2 \to \min_u$, where ρ^2 is a parameter of Tikhonov regularization [9], also has the following shortcomings: 1) it produces a very smooth solution, when the true solution is not smooth; 2) if data are destroyed by spikes, the obtained solution would be bad.

We propose to solve (4) on the basis of the Method of Minimum Duration [12]. Here we change the term "duration" by the term "spatial extent" [13] and use the concept of "spatial extent" two times, namely, in an original space (which is the space of solution or sources) and in a conjugate space (which is the space of equations discrepancy or measurements). Therefore we call this approach as a "Dual Method of Minimum Spatial Extent (DMMSE)" and formulate it as optimization problem

$$f(u) = D_1[Au - b] + \gamma^2 D_2[u] \to \min_u, \quad (5)$$

where $f(u)$ is the objective function of u; term $D_1[Au - b]$ denotes the measure of "spatial extent" of discrepancy $Au - b$; term $D_2[u]$ denotes the measure of "spatial extent" of solution u [14]; γ^2 is a parameter of "external regularization" [4-6].

3. Dual Method Of Minimum Spatial Extent

3.1. Implementation

Idea of DMMSE, presented by (5), consists in the requirement to minimize the spatial extent of solution discrepancy as well as to minimize the spatial extent of solution. To implement the concept of "spatial extent" various approaches can be used [13-17]. Further we use a logarithmic approach to define the "spatial extent" in a form of myriad functional [15]. Then, using (5), for the discrete case we have

$$f(u_1,...,u_N) = c_1 \sum_{j=1}^{J} \ln[1 + |\sum_{n=1}^{N} a_{jn} u_n - b_j|^2 / \alpha_1^2] + \\ + \gamma^2 c_2 \sum_{n=1}^{N} \ln[1 + |u_n|^2 / \alpha_2^2] \to \min_{u_1,...,u_N} \quad (6)$$

where $c_1 = 1/\ln[1 + 1/\alpha_1^2]$; $c_2 = 1/\ln[1 + 1/\alpha_2^2]$; a_{jn} is the element of matrix A; α_1 and α_2 are the parameters of "internal regularization."

Note, if $\alpha_1^2 >> |\sum_{n=1}^{N} a_{jn} u_n - b_j|^2$ for all j, then the first term of objective function (6) tends to the quadratic term, and, consequently, we have the problem [6], that for $\alpha_1^2 >> 1$ can be written as

$$f(u_1,...,u_N) = \sum_{j=1}^{J} |\sum_{n=1}^{N} a_{jn} u_n - b_j|^2 + \\ + \gamma^2 c_2 \sum_{n=1}^{N} \ln[1 + |u_n|^2 / \alpha_2^2] \to \min_{u_1,...,u_N} \quad (7)$$

This is a problem of least squares with non-quadratic regularization term.

Also note, if $\alpha_2^2 >> |u_n|^2$ for all n, then the second term of (6) tends to the quadratic term, corresponded to Tikhonov smoothing [9], that for $\alpha_2^2 >> 1$ can be written as

$$f(u_1,...,u_N) = c_1 \sum_{j=1}^{J} \ln[1 + |\sum_{n=1}^{N} a_{jn} u_n - b_j|^2 / \alpha_1^2] + \\ + \gamma^2 \sum_{n=1}^{N} |u_n|^2 \to \min_{u_1,...,u_N} \quad (8)$$

This is a nonlinear (non-quadratic) minimization problem with quadratic regularization term.

If $\alpha_1^2 >> |\sum_{n=1}^{N} a_{jn} u_n - b_j|^2$ for all j and if $\alpha_2^2 >> |u_n|^2$ for all n, then the first and the second terms of objective function (6) tend to the quadratic terms simultaneously. In this case we have the Tikhonov regularization method [9], which (for $\alpha_1^2 >> 1$ and $\alpha_2^2 >> 1$) can be written for the discrete case as a solving of minimization problem

$$f(u_1,...,u_N)=\sum_{j=1}^{J}|\sum_{n=1}^{N}a_{jn}u_n-b_j|^2+ \\ +\gamma^2\sum_{n=1}^{N}|u_n|^2\to\min_{u_1,...,u_N} \quad (9)$$

The method derived by (9) can be also named as a "dual method of least squares," because it states that the weighted (by γ^2) sum of the squares of the solution discrepancy values and of the squares of the solution values should be a minimum.

3.2. Solving Technique

To solve (6) we use a numerical method, based on the conjugate gradient method given by the following calculation scheme:

$$\begin{aligned} &u^{(t+1)}=u^{(t)}+h^{(t)}p^{(t)},\quad t\ge 0;\\ &p^{(0)}=-g^{(0)};\quad t=0;\\ &p^{(t)}=-g^{(t)}+d^{(t-1)}p^{(t-1)},\quad t\ge 1;\\ &d^{(t-1)}=\|g^{(t)}\|^2/\|g^{(t-1)}\|^2;\\ &h^{(t)}=\arg(\min_h \boldsymbol{f}(u^{(t)}+hp^{(t)})), \end{aligned} \quad (10)$$

where t is a number of iteration; $u^{(t)}$ is a solution at the t–th iteration; $h^{(t)}$ is a step along the descent direction $p^{(t)}$ at the t–th iteration; $g^{(t)}$ is a gradient of functional f at the t–th iteration. Repeating basic steps of conjugate gradient method, proposed method has the following features: 1) calculation of functional gradient; 2) solving of one-dimensional minimization problem for choosing the step size along descent direction.

For simplicity of calculations we transform the complex-valued SLAE (4) to the real-valued SLAE. To do this, we rewrite (4) in the form: $A_0u_0=b_0$, where $A_0=\begin{bmatrix}A^R & -A^I\\ A^I & A^R\end{bmatrix}$ is a purely real-valued block matrix, A^R and A^I are real and imaginary parts of complex-valued matrix A; $b_0=[b^R \quad b^I]^T$ is a purely real-valued vector of measurement, which consists of real b^R and imaginary b^I parts of known complex-valued data; $u_0=[u^R \quad u^I]^T$ is a purely real-valued vector of solution, which consists of real u^R and imaginary u^I parts of complex-valued solution.

Further, we examine the gradient calculation and the solving of one-dimensional minimization problem for the particular case of dipole sources with real-valued amplitudes.

If the solution u is a real-valued vector, we have SLAE: $A_1u=b_0$, where $A_1=\begin{bmatrix}A^R\\ A^I\end{bmatrix}$; $u=u^R$; $b_0=\begin{bmatrix}b^R\\ b^I\end{bmatrix}$. From the necessary optimality condition of (6), for this particular case we have a system of N nonlinear equations

$$\begin{aligned} &c_1\sum_{j=1}^{J}\frac{v_j}{v_j^2+\alpha_1^2}a_{jk}+\gamma^2c_2\frac{u_k}{u_k^2+\alpha_2^2}=0\\ &v_j=\sum_{n=1}^{N}a_{jn}u_n-b_j;\qquad k=1,...,N \end{aligned}, \quad (11)$$

where a_{jn} is an element of matrix A_1; b_j is an element of vector b_0; u_k is an element of real-valued vector u. Note that the left side of (11) represents the gradient components of (6) for this particular case.

If we start iteration process (10) with the $u^{(0)}=0$ (i.e. with the zero initial samples of solution $u_k^{(0)}=0$ for all k), from (11) we have the initial gradient as

$$g^{(0)}=-(A_1)^Tw, \quad (12)$$

where vector w consists of the elements

$$w_j=\frac{b_j}{b_j^2+\alpha_1^2};\qquad j=1,...,J, \quad (13)$$

where b_j is an element of vector b_0. It is interest to note that, if $\alpha_1^2>>b_j^2$, from (12)-(13) we have

$$g^{(0)}\approx-(A_1)^Tb_0. \quad (14)$$

where the right side of (14) is the gradient for least squares technique.

One-dimensional minimization problem $h^{(t)}=\arg(\min f(u^{(t)}+hp^{(t)}))$ consists in the choosing of step size h along the descent direction $p^{(t)}$. We propose to solve this problem by using a finite set of "testing steps". This set includes the

steps which we call "steps to nullify the discrepancy of the solution," "steps to nullify the solution," and "step according to Newton method."

"Steps to nullify the discrepancy of the solution" are used to nullify the some sample of the discrepancy and are defined by

$$h_j^d = -(\sum_{n=1}^{N} a_{jn} u_n^{(t)} - b_j) \Big/ (\sum_{n=1}^{N} a_{jn} p_n^{(t)}); \; j = 1,...,J \; . (15)$$

In total, there are J such steps at the t-th iteration.

"Steps to nullify the solution" are used to nullify the some sample of solution. These steps are defined by

$$h_n^s = -u_n^{(t)} / p_n^{(t)}; \quad n = 1,...,N \; . \qquad (16)$$

In total, there are N such steps at the t-th iteration.

Steps (15) and (16) decrease f according to (6).

"Step according to Newton method" is used to minimize (6) under condition that the solution is in the neighborhood of some local minimum of (6). This step is defined by

$$h = -(g^{(t)}, p^{(t)}) / ([c_1 A^T Q_1 A + \gamma^2 c_2 Q_2] p^{(t)}, p^{(t)}) \; , (17)$$

where $Q_1 = diag(..., \frac{[\alpha_1^2 - (v_j^{(t)})^2]}{[\alpha_1^2 + (v_j^{(t)})^2]^2}, ...)$ is a $J \times J$ diagonal matrix depended on α_1^2 and on the discrepancy values $v_j^{(t)} = \sum_{n=1}^{N} a_{jn} u_n^{(t)} - b_j$ at the t-th iteration; and where $Q_2 = diag(..., \frac{[\alpha_2^2 - (u_j^{(t)})^2]}{[\alpha_2^2 + (u_j^{(t)})^2]^2}, ...)$ is a $N \times N$ diagonal matrix depended on α_2^2 and on the solution values $u_n^{(t)}$ at the t-th iteration. In total, there is only one such step at the t-th iteration.

Here is an algorithm for the use of testing steps. We substitute each of these steps, calculated by (15)-(17), into the one-dimensional minimization problem and choose as $h^{(t)}$ such value of step, for which the functional (6) has a minimal value. If h_j^d or h_n^s has the best value, we regenerate the descent direction as $p^{(t)} = -g^{(t)}$. If h has the best value N times in succession, we regenerate the descent direction also. If the best value of step is equal to zero, we stop the iterations.

4. Numerical Simulation

Fig. 2 and Fig. 3 show results of numerical simulation of direct and inverse problems for the case of electric dipoles array, when dipoles are located along a straight line, which is parallel to the measurement line of electric field, and when their electric moments are perpendicular to the measurement plane (as in Fig.1).

To simulate the direct problem, we used (4) with the following parameters: N=J=200; $\Delta x^{(s)} = \Delta x^{(m)} = \Delta x$; $\Delta x/\lambda$=0.1; z/λ =1, amplitudes u_n=1 for n=40, 70, 100, 130, 160 and u_n=0 for other n (Fig.2,a). Complex values b_j were distorted by additive Gaussian noise as well as by random impulses. Standard deviation of Gaussian noise was about 5% of the maximum of absolute value for the real (Fig.2,b) and for the imaginary (Fig.2,c) parts of electric field component, respectively. The amplitudes of impulses were uniformly distributed in [-1,1], and the probability of their appearance is equal to 0.1 . This means that almost ten percent of the data samples are completely destroyed, and we do not know a priori what data samples are incorrect.

Solution of inverse problem by Tikhonov regularization (9) with the optimal value $\gamma^2 = 1$ is presented in Fig.2,d. This solution does not allow to estimate locations and amplitudes of radiation sources correctly.

Solutions of inverse problem by DMMSE with $u^{(0)} = 0$ for various values of regularization parameters are presented in Fig.3.

Fig.3,a shows solution, obtained with the $\alpha_1 = 1$, $\alpha_2 = 1$, $\gamma^2 = 0.1$, when α_1 and α_2 have big values, for which the first and the second terms of functional (6) are close to the quadratic terms. We can see that this solution is similar to the Tikhonov's solution (9) depicted in Fig.2,d and it does not allow correctly estimating the locations and amplitudes of sources.

Fig.3,b shows solution, obtained with the $\alpha_1 = 0.1$, $\alpha_2 = 1$, $\gamma^2 = 0.1$, when α_2 has a big value, for which the second term of functional (6) is close to the quadratic term and, therefore, when we approximately have the problem (8). Here we see that this solution allows avoiding the drawbacks caused due to impulses, but it does not satisfy the requirement of a minimal spatial extent of solution. Therefore, in this case we have to use some additional a priori information about a source waveform that allows estimating the locations and amplitudes of sources.

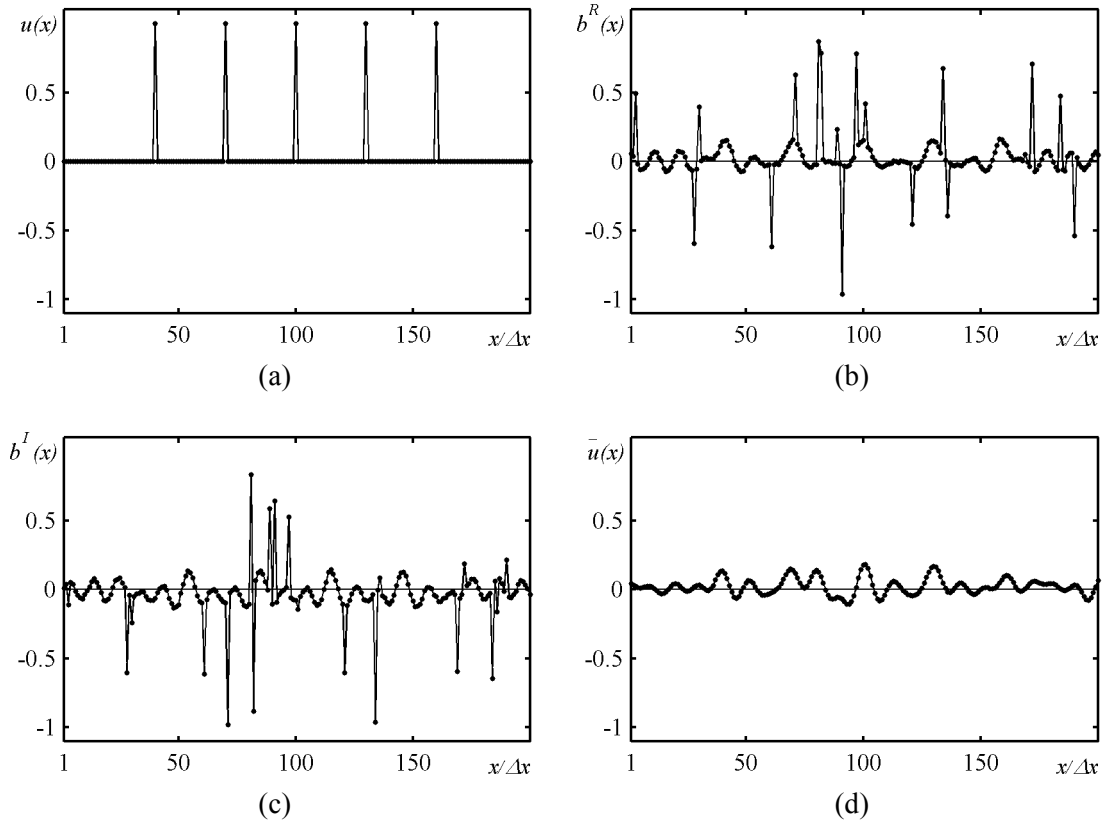

Fig. 2. Spatial distribution of electric field: (a) true real-valued spatial distribution; (b) real and (c) imaginary parts of electric field, distorted by additive and impulsive noises; (d) solution of inverse problem after use of Tikhonov regularization with the optimal value γ^2=1.

Fig.3,c shows solution, obtained with the $\alpha_1 = 1$, $\alpha_2 = 0.1$, $\gamma^2 = 0.1$, when α_1 has a big value, for which the first term of functional (6) is close to the quadratic term and, therefore, when we approximately have the problem (7). As expected, the impulsive noise destroys the solution and leads to the wrong results of estimation.

Finally, the solution by DMMSE with the quasi-optimal values of parameters $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\gamma^2 = 0.1$, obtained after 144 iterations, is presented in Fig.3,d. In this case the graphics of the obtained (Fig.3,d) and the true (Fig.2,a) solutions were almost identical. Note that obtained solution allows to make the correct estimation of the locations and amplitudes of radiation sources using samples which absolute amplitude values are greater than the value of α_2.

We study also the behavior of maximal value of Relative Mean Square Error (RMSE) of solution. Table 1 contains results obtained over 100 realizations of the mixture of Gaussian additive noise and uniformly distributed impulsive noise for the different values of internal and external regularization parameters. The value of RMSE was calculated with the following formula: $\delta = \sqrt{\sum_{n=1}^{N} (\bar{u}_n - u_n)^2 / \sum_{n=1}^{N} u_n^2} \times 100\%$, where $\bar{u}_n$ denotes a sample of obtained solution, and u_n denotes a sample of true solution. The maximal value of RMSE was calculated by use of the maximization operation to the RMSE values obtained over 100 realizations of the mixture of noises for given values of regularization parameters.

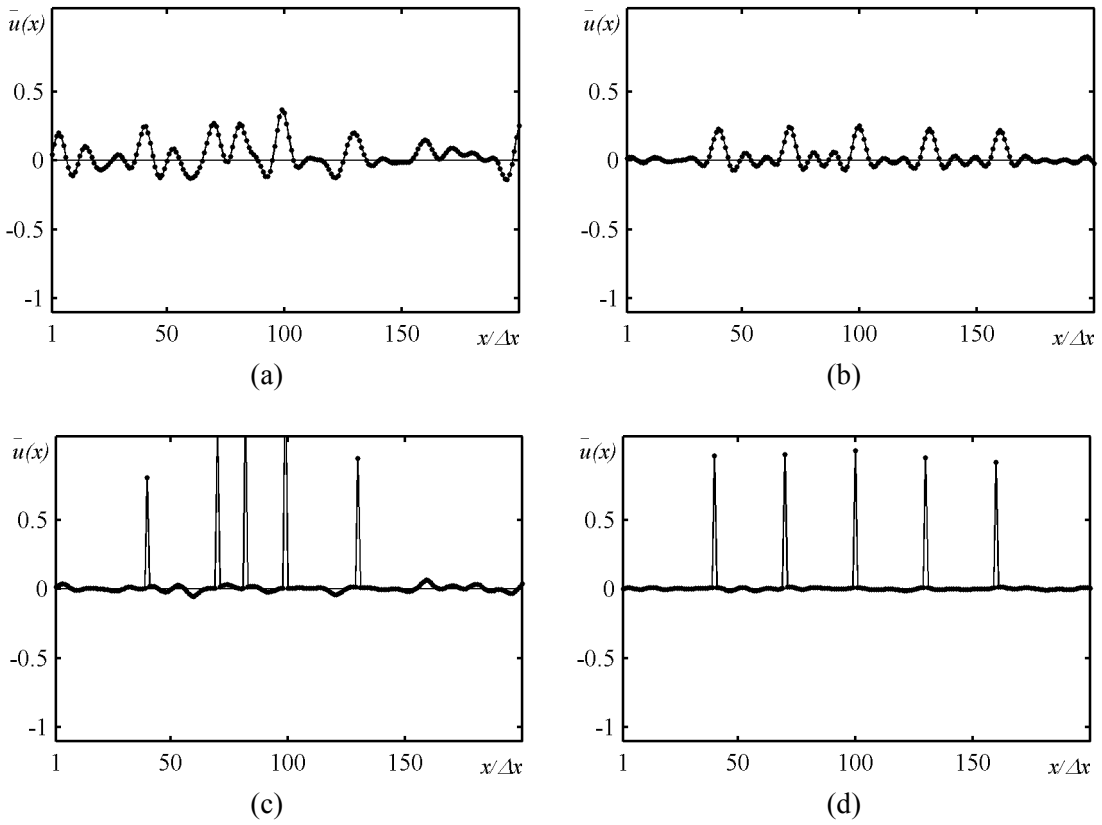

Fig. 3. Spatial distribution after use of DMMSE: (a) with $\alpha_1 = 1$, $\alpha_2 = 1$, $\gamma^2 = 0.1$, when α_1 and α_2 are very big; (b) with $\alpha_1 = 0.1$, $\alpha_2 = 1$, $\gamma^2 = 0.1$, when α_2 is very big; (c) with $\alpha_1 = 1$, $\alpha_2 = 0.1$, $\gamma^2 = 0.1$, when α_1 is very big; and (d) with $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\gamma^2 = 0.1$, which are close to the optimal values.

Table 1 shows that for the case depicted in Fig.3,d maximal value of RMSE equals to 12.68%. It also shows that there exist values of the regularization parameters, for which a smaller value (of RMSE maximal value) is achieved.

Table 1. Maximal value of RMSE (%) obtained over 100 realizations of the mixture of additive and impulsive noises.

$\alpha_1^2 = \alpha_2^2$	$\log_{10}(\gamma^2)$				
	-2	-1.5	-1	-0.5	0
0.1	328.56	189.13	100.44	91.17	94.77
0.01	123.69	16.30	12.68	12.27	101.46
0.001	95.05	7.46	5.05	4.50	78.41
0.0001	145.00	117.10	57.47	56.98	141.18

We can see that the minimum of maximal values of RMSE is approximately equal to the percentage of Gaussian noise in the input data (about 5%). Thus, Table 1 shows the stability degree of obtained solutions and indicates the existence of the optimum values of the regularization parameters.

Note that the large values of RMSE were caused by different reasons. The large values of RMSE for the small values of α_1, α_2 were obtained due to violation of the algorithm convergence. But the large values of RMSE for the large values of α_1, α_2 were obtained due to the shifting of the global minimum with respect to the true solution. This is due to the fact that for small values of α_1, α_2 the location of the global minimum is close to the true value, but there are many local minimums. The number of local minimums becomes greater, the values of

α_1, α_2 become the smaller. Therefore, the process of convergence to a global minimum is violated, and the convergence to a local minimum gives incorrect results.

5. Conclusion

DMMSE is the generalization of least squares technique with Tikhonov regularization. It allows estimating the location and amplitude of radiation sources from the near-field measurements. DMMSE is stable to additive and impulse noise, and it allows to solve the estimation problem under conditions when locations of sources are unknown a priori. It is shown that there are optimal values of the regularization parameters, for which obtained solution practically coincides with the desired solution. Numerical simulation confirmed an advantage of the proposed method compared to the traditional method of least squares using quadratic Tikhonov regularization. Proposed implementation of the method based on the conjugate gradient method and original algorithm for solving of one-dimensional optimization problem allows us to achieve a high speed of data processing with the ability to perform it real-time. Our further research is focused on an adaptive approach to select the regularization parameters.

References

1. Yaghjian A.D.: *An Overview of Near-Field Antenna Measurements*. In: IEEE Transactions on Antennas and Propagation, Vol. 34, Jan. 1986, p. 30-44.
2. Cano F., Sierra-Castaner M., Burgos S., Besada J. L.: *Applications of sources reconstruction techniques: theory and practical results*. In: Proceedings of the 4th European Conference on Antennas and Propagation (EuCAP), Barcelona, 2010 , p. 1–5.
3. Martini E., Breinbjerg O., Maci S.: *Reduction of Truncation Errors in Planar Near-Field Aperture Antenna Measurements Using the Gerchberg-Papoulis Algorithm.* In: IEEE Transactions on Antennas and Propagation, Vol. 56, Nov. 2008, p. 3485-3493.
4. Vovk S. M., Borulko V. F.: *Using the method of minimum extension to extrapolation of antenna measurements*. In: Proceedings of the 10th International Conference on Mathematical Methods in Electromagnetic Theory, Dniepropetrovsk, Ukraine, 2004, p. 159-161.
5. Vovk S. M., Borulko V. F.: *Restoration of point sources located along a perpendicular to a plane of measurements by the extension minimum method.* In: Proceedings of the 11th International Conference on Mathematical Methods in Electromagnetic Theory, Kharkiv, Ukraine, 2006, p. 147-149.
6. Borulko V. F., Vovk S. M.: *Determining sources in the array of electric dipoles.* In: Proceedings of the International Conference on Mathematical Methods in Electromagnetic Theory, Kyiv, Ukraine, 2010, p. 1-4.
7. Forsythe G. E., Malcolm M. A., Moler C. B.: *Computer Methods for Mathematical Computations*, New York, Prentice-Hall, 1977.
8. Co T. B.: *Methods of Applied Mathematics for Engineers and Scientists*, New York, Cambridge University Press, 2013.
9. Tikhonov A. N., Arsenin V. Y.: *Solutions of ill-posed problems*, New York, Wiley, 1977.
10. Huber P. J.: *Robust statistics*. New York: John Wiley and Sons, 1981.
11. Neff H. P.: *Introductory Electromagnetics,* New York, Wiley, 1991.
12. Vovk S. M., Borul'ko V. F.: *A minimum-duration method for recovering finite signals*. In: Radioelectronics and Communications, vol. 34, no. 8, 1991, p. 67-69.
13. Borulko V., Vovk S.: *Image Processing by Minimum Extent Method*. In: Proceedings of the International Conference on Pattern Recognition and Information Analysis, Minsk, Republic of Belarus, 1995, p. 71-72.
14. Vovk S. M., Borul'ko V. F.: *Statement of a problem of definition of linear signals parameters in quasinormed space*. In: Radioelectronics and Communications, Vol. 53, no. 7, 2010, p. 367-375.
15. Vovk S. M., Borulko V. F.: *Myriad and meridian functionals as extreme forms of smoothed quasinorms.* In: Proceedings of the 14th International Conference on Mathematical Methods in Electromagnetic Theory, Kharkiv, Ukraine, 2012, p. 320-323.
16. Vovk S., Borulko V.: *Family of generalized Demidenko functionals for robust estimation*. In: Proceedings of the 17th International Seminar/Workshop on Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory, DIPED'12, Tbilisi, Georgia, September 2012, p. 151–154.
17. Vovk S., Borulko V.: *Spectrum extrapolation by using combined root-Demidenko family of quasi-duration functionals*. In: Proceedings of the 8th International Kharkov Symposium on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves, MSMW'13, Kharkov, Ukraine, June 2013, p. 461–463.