

# CLOSED-FORM SOLUTION OF A THREE-PHASE VOLTAGE-SOURCE INVERTER FEEDING A THREE-PHASE INDUCTION MOTOR

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**Abstract:** In this paper a closed-form solution of the phase current of a three-phase, voltage-source inverter (VSI) fed induction motor (IM) is presented. Instead of using numerical solution methods, explicit analytical expressions of the steady-state current of the motor are derived.

To validate the approach presented in this paper the results obtained are compared with previously published numerical results dealing with static R-L loads; considering the induction motor as equivalent to a R-L load, and the two sets of results are found to be almost identical.

**Keywords:** Closed-form solution, Voltage-source inverter, Induction motor, Steady-state analysis, numerical solution.

## 1. Introduction

The voltage-source inverters are commonly used in speed control of three-phase induction motors from a DC supply. By controlling the switching elements in the voltage-source inverter the applied voltage to the induction motor can be controlled.

A lot of attention was given to voltage-source inverters, control of three-phase induction motors and voltage-source inverters /induction motor systems [1-21].

In previous investigations, numerical methods were used to obtain the transient performance of the system [22-27], and the steady-state performance was obtained

by passing through the transient performance first, and were not obtained directly.

In this paper, direct analytical closed-form expressions, at steady-state and at fundamental frequency, for the currents of the VSI/IM system are derived.

To check the validity of the presented approach the results obtained are compared with previously published results for static R-L loads by considering the induction motor, at any speed, as corresponding to a static R-L load.

## 2. Method of analysis

At steady state, and any speed, the equivalent circuit per phase of the induction motor is as shown in Fig. 1. This equivalent circuit of the induction motor can be presented by an equivalent series R-L load. If the corresponding equivalent resistance is  $R_e$ , and the corresponding equivalent reactance is  $X_e$ , then:

$$R_e = R_1 + \frac{X_m^2 R_2' s}{R_2'^2 + s^2 (X_m + X_2')^2} \quad (1)$$

and

$$X_e = X_1 + \frac{R_2'^2 + s^2 X_2'^2 (X_m + X_2')^2}{R_2'^2 + s^2 (X_m + X_2')^2} \cdot X_m \quad (2)$$

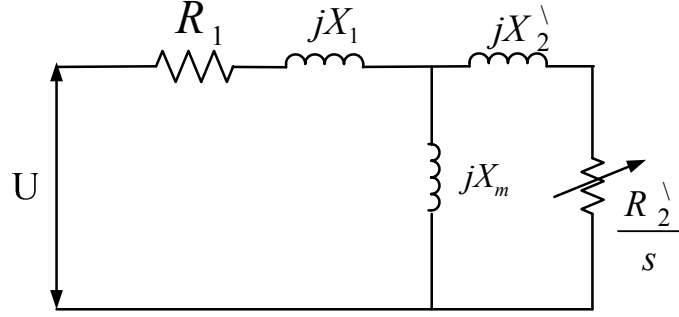


Fig. 1. Induction motor equivalent circuit.

If the phase angle of the motor at any slip,  $s$ , is assumed to be  $\varphi$ , then:

$$\tan(\varphi) = X_e / R_e \quad (3)$$

A voltage-source inverter feeding a three-phase induction motor is shown in Fig. 2. In this circuit the switching elements (S) may be either thyristors or transistors. Each switching element will conduct for  $180^\circ$  during a cycle.

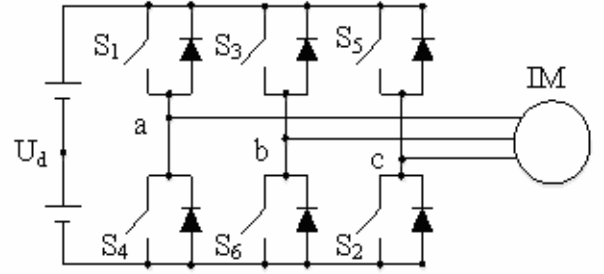


Fig. 2. Voltage-source inverter/induction motor system.

The differential equation of the line current of the motor for line (a) is:

$$u_a(\omega t) = R_e i_a + L_e \frac{di_a}{dt} \quad (4)$$

where  $R_e$  is the input equivalent resistance of the equivalent circuit of the induction motor and  $L_e$  is the inductance corresponding to  $X_e$  at fundamental frequency.

The inverter six-step voltage waveform,  $u_a$ , applied to phase (a) of the motor is shown in Fig. 3, together with a typical waveform of the phase current [1, 7, 22, 28].

Thus, solution of eqn. (4) in the period  $0 \leq \omega t \leq \pi/3$ , in which  $u_a(\omega t) = U_d/3$ , will be:

$$i_{a1}(\omega t) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}]e^{-\omega t \cot \phi} \quad (5)$$

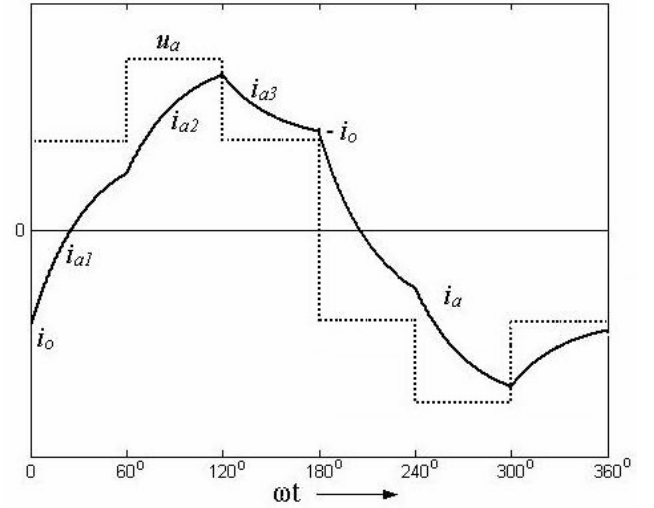


Fig. 3. Voltage-source inverter output phase (a) voltage and current.

where:

$i_o$  is the initial value of  $i_a(\omega t)$  at  $\omega t = 0$ , and its derivation will be given in Section (2.1), and its expression is as given in eqn. (14).

$U_d$  is the inverter input DC voltage.

$\phi$  phase angle of the induction motor at fundamental frequency, i.e.  $\phi = \tan^{-1}(\omega L_e/R_e)$ .

Eqn. (5) can be rewritten as:

$$i_{a1}(\omega t) = k_1 + \left[ \frac{k_1(k_2 - 2)}{k_2^2 - k_2 + 1} \right] e^{-\omega t \cot \phi} \quad 0 \leq \omega t \leq \pi/3 \quad (6)$$

where:  $k_1 = U_d / (3R_e)$  and  $k_2 = e^{-(\pi/3) \cot \phi}$

For the period  $\pi/3 \leq \omega t \leq 2\pi/3$ , in which  $u_a(\omega t) = 2 U_d / 3$ , Fig. 3, solution of eqn. (4) is obtained as:

$$i_{a2}(\omega t) = \frac{2U_d}{3R_e} + [i_{a1}(\pi/3) - \frac{2U_d}{3R_e}] e^{-(\omega t - (\pi/3)) \cot \phi} \quad \pi/3 \leq \omega t \leq 2\pi/3 \quad (7)$$

where  $i_{a1}(\pi/3)$  is obtained from eqn. (5) with  $\omega t = \pi/3$ . Thus:

$$i_{a1}(\pi/3) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}] e^{-(\pi/3) \cot \phi}$$

Therefore, eqn. (7) becomes:

$$i_{a2}(\omega t) = \frac{2U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}] e^{(\pi/3) \cot \phi} - \frac{U_d}{3R_e} e^{-\omega t \cot \phi} \quad \pi/3 \leq \omega t \leq 2\pi/3 \quad (8)$$

Eqn. (8) can be rewritten as:

$$i_{a2}(\omega t) = 2k_1 - \left[ \frac{k_1(k_2 + 1)}{k_2(k_2^2 - k_2 + 1)} \right] e^{-\omega t \cot \phi} \quad \pi/3 \leq \omega t \leq 2\pi/3 \quad (9)$$

For the period  $2\pi/3 \leq \omega t \leq \pi$ , in which  $u_a(\omega t) = U_d/3$ , Fig. 3, solution of eqn. (4) becomes:

$$i_{a3}(\omega t) = \frac{U_d}{3R_e} + [i_{a2}(2\pi/3) - \frac{U_d}{3R_e}] e^{-(\omega t - (2\pi/3)) \cot \phi} \quad 2\pi/3 \leq \omega t \leq \pi \quad (10)$$

where  $i_{a2}(2\pi/3)$  is obtained from eqn.(8) at  $\omega t = 2\pi/3$ . Thus:

$$i_{a2}(2\pi/3) = i_o e^{-(2\pi/3) \cot \phi} + \frac{U_d}{3R_e} [2 - e^{-(\pi/3) \cot \phi} - e^{-(2\pi/3) \cot \phi}]$$

$$i_{a2}(2\pi/3) = i_o e^{-(2\pi/3) \cot \phi} + \frac{U_d}{3R_e} [2 - e^{-(\pi/3) \cot \phi} - e^{-(2\pi/3) \cot \phi}]$$

Therefore, eqn. (10) becomes:

$$i_{a3}(\omega t) = \frac{U_d}{3R_e} + [i_o e^{-(2\pi/3) \cot \phi} + \frac{U_d}{3R_e} (2 - e^{-(\pi/3) \cot \phi} - e^{-(2\pi/3) \cot \phi}) - \frac{U_d}{3R_e}] e^{-(\omega t - 2\pi/3) \cot \phi}$$

$$2\pi/3 \leq \omega t \leq \pi \quad (11)$$

Eqn. (11) can be rewritten as:

$$i_{a3}(\omega t) = k_1 + \left[ \frac{k_1(1 - 2k_2)}{k_2^2(k_2^2 - k_2 + 1)} \right] e^{-\omega t \cot \phi} \quad 2\pi/3 \leq \omega t \leq \pi \quad (12)$$

Thus, during the first positive half cycle of the phase voltage applied to the motor, Fig.(3), the expression of the waveform of the steady-state current of phase (a) of the motor will be:

$$i_a = i_{a1} \quad 0 \leq \omega t \leq \pi/3$$

$$i_a = i_{a2} \quad \pi/3 \leq \omega t \leq 2\pi/3$$

$$i_a = i_{a3} \quad 2\pi/3 \leq \omega t \leq \pi$$

$i_{a1}$ ,  $i_{a2}$  and  $i_{a3}$  are obtained from eqns. (6), (9)

and (12) respectively.

### 2.1 Determination of the initial current $i_o$

Since, the current of  $i_{a1}(\omega t)$  at  $\omega t = 0$  is  $i_o$  and the current of  $i_{a3}(\omega t)$  at  $\omega t = \pi$  is  $(-i_o)$ , Fig. 3, thus:

$$i_{a3}(\pi) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e} e^{(\pi/3)\cot\phi} + \frac{U_d}{3R_e} e^{(2\pi/3)\cot\phi} - \frac{U_d}{3R_e}] e^{-\pi\cot\phi} = -i_o$$

Thus:

$$i_o = \frac{\frac{U_d}{3R_e} (e^{-\pi\cot\phi} + e^{-(2\pi/3)\cot\phi} - e^{-(\pi/3)\cot\phi} - 1)}{(1 + e^{-\pi\cot\phi})} \quad (13)$$

Eqn. (13) can be rewritten as:

$$i_o = \frac{k_1(k_2^3 + k_2^2 - k_2 - 1)}{(k_2^3 + 1)} \quad (14)$$

With the initial current  $i_o$  determined using eqn. (13) or eqn. (14), for certain phase angle of the motor,  $\phi$ , direct and explicit form expressions of the steady-state induction motor current can be obtained.

### (2.2) Determination of the motor rms Current

The rms current can be obtained, Fig. (3), from:

$$I_a = \left[ \frac{1}{\pi} \left\{ \int_0^{\pi/3} i_{a1}^2(\omega t) d\omega t + \int_{\pi/3}^{2\pi/3} i_{a2}^2(\omega t) d\omega t + \int_{2\pi/3}^{\pi} i_{a3}^2(\omega t) d\omega t \right\} \right]^{0.5} \quad (15)$$

Using the expressions of  $i_{a1}$ ,  $i_{a2}$  and  $i_{a3}$  given in eqns. (6), (9) and (12) respectively, the rms current of the motor ( $I_a$ ) can be obtained from eqn. (15) as:

$$I_a = \left[ \frac{1}{\pi} \left\{ 2\pi k_1^2 + 6k_1 i_o \tan(\phi) - \frac{3i_o^2 \tan(\phi)(k_2^2 - k_2 + 1)}{(k_2^2 - 1)} \right\} \right]^{0.5} \quad (16)$$

### (2.3) Input equivalent resistance and reactance of the motor

For a certain value of the phase angle of the induction motor,  $\phi$ , and using eqns.(1) and (2):

$$\tan(\phi) = X_e / R_e$$

Thus :

$$X_e = R_e \tan(\phi)$$

and the following expression relating the parameters of the motor with slip,  $s$ , and  $\phi$  can be obtained as:

$$A s^2 + B s + C = 0 \quad (17)$$

where:

$$A = (X_m + X_2)(X_l X_m + X_l X_2 + X_2 X_m - (R_l X_m + R_l X_2) \tan(\phi))$$

$$B = X_m^2 R_2 \tan(\phi)$$

$$C = (R_2)^2 [X_l + X_m - R_l \tan(\phi)]$$

The slip,  $s$ , can be obtained from the solution of eqn. (17) as:

$$s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

### (2.4) Relationship between motor and static R-L load currents

The impedance of the motor can be expressed by :

$$Z_m = R_e \sqrt{1 + (\tan \phi_m)^2} \quad (18)$$

and that of the static R-L load can be expressed by:

$$Z = R \sqrt{1 + (\tan \phi_{RL})^2} \quad (19)$$

Thus, the relationship between the rms current of the motor,  $I_m$ , and the rms current of the static R-L load,  $I_{RL}$ , for the same applied voltage, will be:

$$\frac{I_m}{I_{RL}} = \frac{R \sqrt{1 + \tan(\phi_{RL})^2}}{R_e \sqrt{1 + \tan(\phi_m)^2}} \quad (20)$$

Thus, if the phase angle of the motor,  $\phi_m$ , is equal to the phase angle of the static R-L load,  $\phi_{RL}$ , the relationship between  $I_m$  and  $I_{RL}$  will be :

$$\frac{I_m}{I_{RL}} = \frac{R}{R_e} \quad (21)$$

i.e. at the same phase angle:

$$I_m = \frac{R}{R_e} I_{RL} \quad (22)$$

### 3. Results

In order to obtain the results, for the presented approach, for the current of phase (a) of the motor, as an example, MATLAB programming package is used to obtain the results at steady-state and at fundamental frequency when the three-phase voltage-source inverter is feeding the three-phase induction motor whose parameters are given in the Appendix.

To check the validity of the proposed approach, the obtained results are compared with previously published results obtained for static R-L loads [28]. In reference [28] results were obtained numerically and analytically when a VSI is feeding a static R-L load.

Fig. 4 shows the instantaneous values of the phase current  $i_a$  for the case of the motor at a slip,  $s$ , that will give a phase angle  $\varphi = 30^\circ$  and an input impedance of  $Z_m = 10 \Omega$ . In this case eqn. (1) is used to obtain the corresponding slip of the motor, which was obtained as  $s=0.0413$ .

The instantaneous current of the motor is compared with the current of a static R-L load whose  $\varphi_{RL} = 30^\circ$ , and its impedance is  $Z = 10 \Omega$ . It is evident from this figure, Fig. 4, that the results of the motor current and the static R-L load current are almost identical, which proves the validity of the approach presented.

The rms current of the motor at any phase angle,  $\varphi_m$ , can be obtained using the rms current value of the static R-L load, at the same phase angle and applied voltage, employing eqn.(22).

In eqn. (22) the equivalent resistance of the motor  $R_e$  at certain slip is required.

To obtain the slip corresponding to a certain phase angle,  $\varphi_m$ , of the motor, Fig. 5, which relates the motor power factor with its slip, is used. Fig. 5 is obtained using eqns. (1), (2) and (3). For the range of the slip considered (0 to 1), for each value of the slip,  $s$ , eqns. (1) and (2) are used to obtain the corresponding values of  $R_e$  and  $X_e$ , and the corresponding phase angle is obtained from eqn. (3). For a phase angle for the static R-L load as that of the motor and for any value of its impedance,  $Z$ , and using the same applied voltage as that applied to the motor, its rms current can be obtained. Thus, to obtain the motor rms current at the same phase angle as that of the static R-L load and at the same applied voltage, eqn. (22) can be used directly.

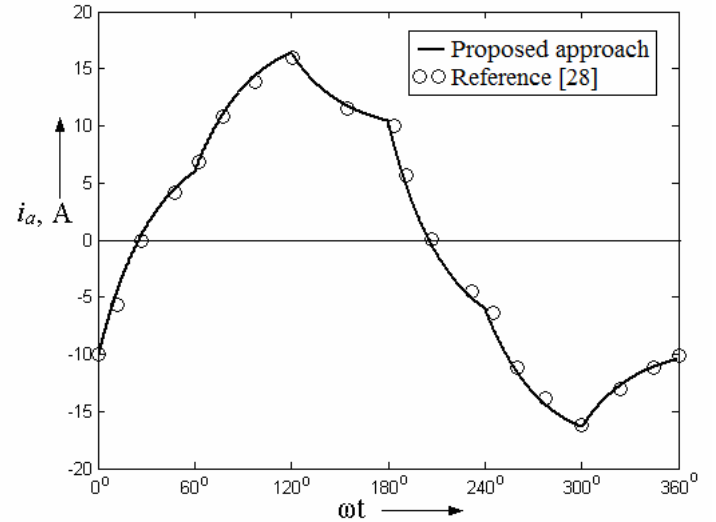


Fig. 4. Phase (a) current waveform.

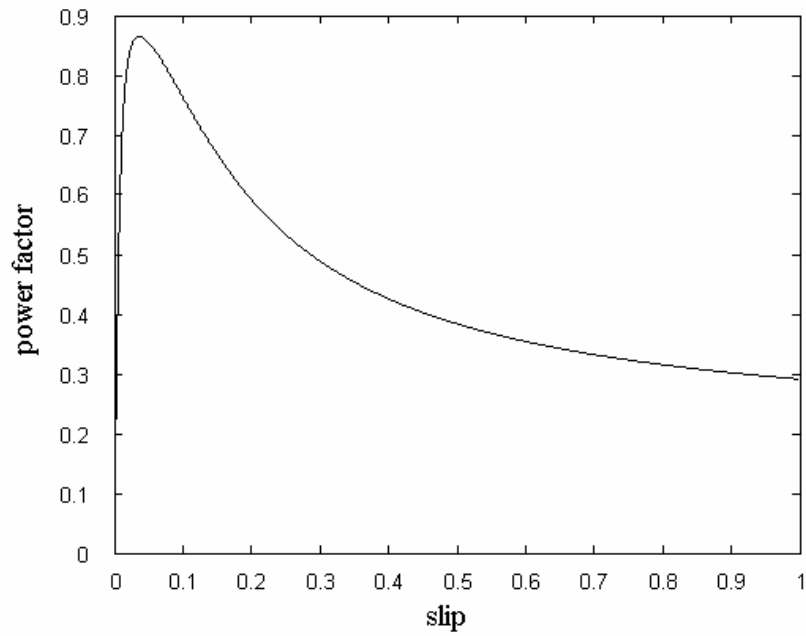


Fig. 5. Motor power factor/ slip characteristic.

Fig. 6 shows a comparison between the rms values of the currents of the motor and of the static R-L load, when fed from three-phase voltage-source inverters, for different values of phase angles. From this figure it is evident the two sets of results are in very close agreement.

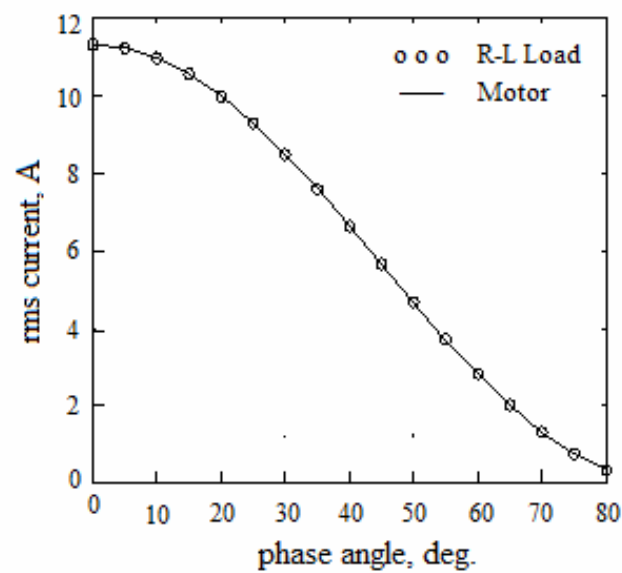


Fig. 6. Comparison between rms values of the motor current and the static R-L current.

#### 4. Conclusions

In this paper, closed-form analytical solutions for currents at steady-state and fundamental frequency for the three-phase induction motor fed from a three-phase voltage-source inverter are presented. These explicit closed-form expressions of the motor current can be used to obtain the performance characteristics of the system.

The results obtained using these analytical closed-form expressions were compared with those obtained analytically and numerically in previously published reference dealing with three-phase static R-L load fed from a three-phase voltage-source inverter, and the two sets of results were in very close agreement.

#### List of symbols

$R_1, R_2$	induction motor stator and rotor resistances per phase respectively referred to stator.
$X_1, X_2$	induction motor stator and rotor leakage reactances per phase respectively referred to stator.
$X_m$	induction motor magnetizing reactance per phase.
$U$	induction motor terminal voltage per phase.
$s$	induction motor slip.
$\omega$	fundamental angular frequency of the supply.

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## Appendix

The parameters of the induction motor used in this system are [12]:

Three-phase induction motor, 7500-W, 400-V, 50-Hz, 16-A, 4-pole with the following parameters:

$$R_1 = 0.6 \, \Omega, X_1 = 0.9425 \, \Omega, R_2' = 0.4 \, \Omega, X_2' = 2.325 \, \Omega, X_m = 37.7 \, \Omega.$$