

MAXWELL EQUATIONS CONTRADICT THE “NEGATIVE REFRACTION” CONCEPT

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Abstract The paper analyses the considerations that led decades ago to the hypothesis of “materials of left hand”. It is shown that they are in contradiction with Maxwell equations. All documentation is at the macroscopic level of classical electrodynamics.

1. Introduction.

A few decades ago the unusual idea that the refraction index of materials n :

$$n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} > 0 \quad (1)$$

may be defined also for negative electric permittivity and negative magnetic permeability [1]:

$$n' = \sqrt{\frac{(-\epsilon)(-\mu)}{\epsilon_0\mu_0}} \quad (2)$$

If formally (mathematically) this seems possible is it also physically valid?

In search for the consequences of this hypothesis, Ref[1] made use of Maxwell equation in a homogenous isotropic, charge-less and current-less medium (SI units are used):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad (4)$$

with the constitutive relations

$$\vec{B} = \mu \vec{H}; \quad \vec{D} = \epsilon \vec{E} \quad (5)$$

with which (3)-(4) becomes

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}; \quad \mu > 0 \quad (3')$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}; \quad \epsilon > 0 \quad (4')$$

For plane electromagnetic waves the above equations lead to:

$$\vec{k} \times \vec{E} = \omega \mu \vec{H} \quad (6)$$

$$\vec{k} \times \vec{H} = -\omega \epsilon \vec{E} \quad (7)$$

where ω is the angular pulsation and:

$$\vec{k} = k \vec{u}_k = \frac{2\pi}{\lambda} \vec{u}_k \quad (8)$$

is the wave vector.

From (6)-(7) it was inferred in [1] that while for $\epsilon > 0, \mu > 0$ the $\vec{E}, \vec{H}, \vec{k}$ vectors form (in this order) a “right triplet” (Fig.1a), for $\epsilon < 0, \mu < 0$ when eqns (6)-(7) become:

$$\vec{k} \times \vec{E} = -\omega |\mu| \vec{H}; \quad |\mu| > 0 \quad (9)$$

$$\vec{k} \times \vec{H} = \omega |\epsilon| \vec{E}; \quad |\epsilon| > 0 \quad (10)$$

the new triplet $\vec{E}, \vec{H}, \vec{k}' = -k \vec{u}_k$ is of “left hand” (Fig 1.b):

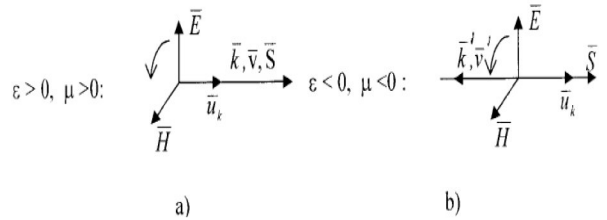


Fig.1 $\vec{E}, \vec{H}, \vec{k}$ triplet a) of “right hand”, b) of “left hand”

On the other hand, as the Poynting vector \vec{S} :

$$\vec{S} = \vec{E} \times \vec{H} \quad (11)$$

always makes with \vec{E}, \vec{H} a “right hand” triplet it may be inferred that when $\epsilon < 0, \mu < 0$ the wave vector (and thus the phase speed) is opposite to the direction of energy flux propagation. In “left hand” substances the phase velocity is opposite to energy flux [1], Fig.2.b.

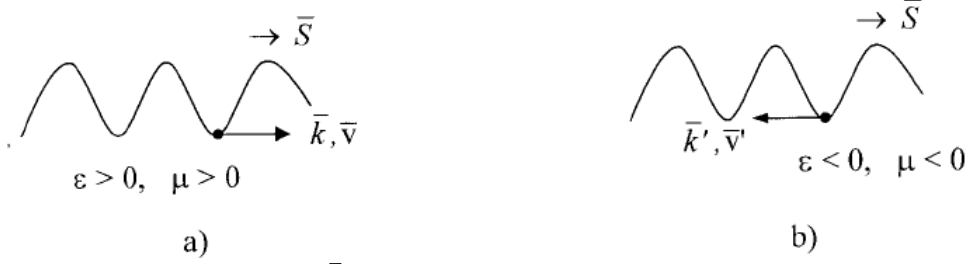


Fig.2 Poynting vector \vec{S} and phase velocity \vec{v} (\vec{v}), a) for $\epsilon > 0, \mu > 0$, b) for $\epsilon < 0, \mu < 0$

For the above reasons, the hypothetical materials with $\epsilon < 0, \mu < 0$ have been called of “left hand” (LHM-left hand materials) as in [1]; other researchers considered this paradoxical situation as corresponding to “left handed light” (LHL) [2].

The potential consequences of such electromagnetic materials (EM) would be quite strange (exotic) [1,3]), such as the inverse Doppler and Vavilov-Cerenkov effects, the possibility to become invisible (invisibility cloak) etc.

About the above claims a few objections could be made. The main one is that from a physical point of view it is impossible that the phase speed (or the wave vector) were opposite to the electromagnetic energy flux (“evanescent waves”), because the wave itself carries the respective energy. This latter physical “postulate” is independent of the nature of the material and demands a critical reassessment of the “left hand” materials concept. We will show that in the rationale of [1] a subtle error slipped in and perpetuated until today. Our analysis will be based also on Maxwell equations, but as their integral forms contains both “rotors” and “fluxes” of \vec{E} and \vec{H} vectors, we first reiterate a few observations on vectorial elements association rules because they will turn out of crucial importance later on in our study.

2. Referential directions and their association rules

For the line integral, the vectorial field \vec{F} circulation case included, the referential direction of the scalar variable represented by the integral [4] is the direction of the curve vectorial element, $d\vec{l}$ (Fig.3a). It is evident that operating with an opposite referential direction $d\vec{l}$, the scalar variable (electromagnetic force) will change sign:

$$\oint_{\Gamma} \vec{E} \cdot d\vec{l} = -\oint_{\Gamma} \vec{E} \cdot d\vec{l}; \quad \oint_{\Gamma} \vec{H} \cdot d\vec{l} = -\oint_{\Gamma} \vec{H} \cdot d\vec{l} \quad (12)$$

The flux of \vec{F} may be defined with respect to an open surface S_{Γ} (enclosed by the curve Γ) or with respect to a closed surface and its referential direction is given by the surface vectorial element $d\vec{s}$ (Fig.3b). As electrodynamics makes use of Stokes theorem, when the “rotor” and the “flux” through open surfaces occur simultaneously:

$$\oint_{\Gamma} \vec{F} \cdot d\vec{l} = \iint_{S_{\Gamma}} (\nabla \times \vec{F}) \cdot d\vec{s}, \quad (13)$$

The referential directions of these variables have to be correlated, that is, adequately associated.

In principle, there are two possibilities: right hand rule association (Fig.3b) or left hand rule association (Fig.3c).

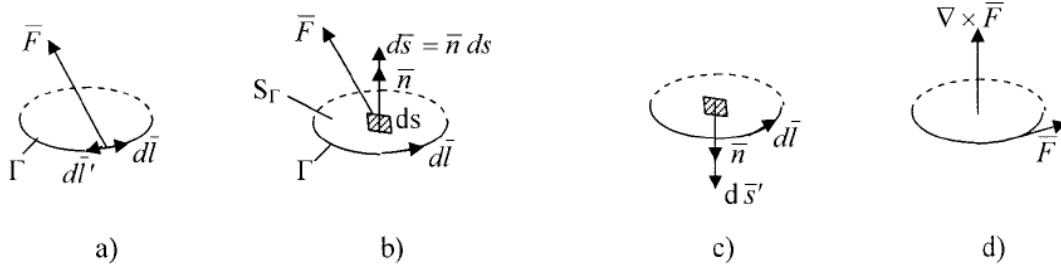


Fig.3 Referential directions and their association a) curve vectorial element $d\vec{l}$, b) surface vectorial element $d\vec{s}$, c) left hand rule association, d) the rotor’s “right hand” rule

The rule for which the terms in (13) are positive has been chosen. Or, as by definition \vec{F} and $\nabla \times \vec{F}$ are associated by the right hand rule (Fig.3d), it follows that the geometrical vectors \vec{dl} and \vec{ds} have to be associated likewise. The literature calls the unitary vector \vec{n} associated this way with \vec{dl} [5] as the "positive normal". In contrast, operating –for example – with $\vec{ds} = -\vec{ds}$ ("left hand" rule or "negative normal", Fig.3c) eqn (13) would become:

$$\oint_{\Gamma} \vec{F} \cdot \vec{dl} = \int_{S_r} (\nabla \times \vec{F}) \cdot \vec{ds}' = - \int_{S_r} (\nabla \times \vec{F}) \cdot \vec{ds} \quad (14)$$

More generally (and reciprocally) changing the sign of only one term in eqn.(13) is equivalent to shifting to the opposite referential directions associations rules.

3. Some notes on Maxwell equations in integral form

For the free space the Maxwell evolutionary equations in integral form write:

$$\oint_{\Gamma} \vec{E} \cdot \vec{dl} = - \int_{S_r} \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad (15)$$

$$\oint_{\Gamma} \vec{H} \cdot \vec{dl} = \int_{S_r} \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} \quad (16)$$

where both the geometrical vectorial elements (Fig.4a) and the field vectors are associated by same "right hand" rule (Fig.4b).

It should be noticed that by multiplying (15),(16) with (-1) we unfold the same laws; that is the pairs of vectors (geometrical and of field ones) keep their "right hand" rule association (Fig.4c):

$$\oint_{\Gamma} (-\vec{E}) \cdot \vec{dl} = \int_{S_r} \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad (17)$$

$$\oint_{\Gamma} (-\vec{H}) \cdot \vec{dl} = \int_{S_r} (-\frac{\partial \vec{D}}{\partial t}) \cdot \vec{ds} \quad (18)$$

However, if we change the sign of only one term in one equation, for example we eliminate the sign (-) in Faraday's law (15) we obtain:

$$\oint_{\Gamma} \vec{E} \cdot \vec{dl} = \int_{S_r} \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad (19)$$

which will change also the initial association rule of field vectors (the one of geometrical vector remains unchanged, Fig.5a); \vec{E} and $-\frac{\partial \vec{B}}{\partial t}$ will form then a

"left hand" rule system (Fig.5b) As, however, in both equations (15)-(16), the field vectors have to be associated according to same rule, the sign (-) in (15) is propagated "automatically" in Ampere's law [6]:

$$\oint_{\Gamma} (\vec{H}) \cdot \vec{dl} = \int_{S_r} (-\frac{\partial \vec{D}}{\partial t}) \cdot \vec{ds} \quad (20)$$

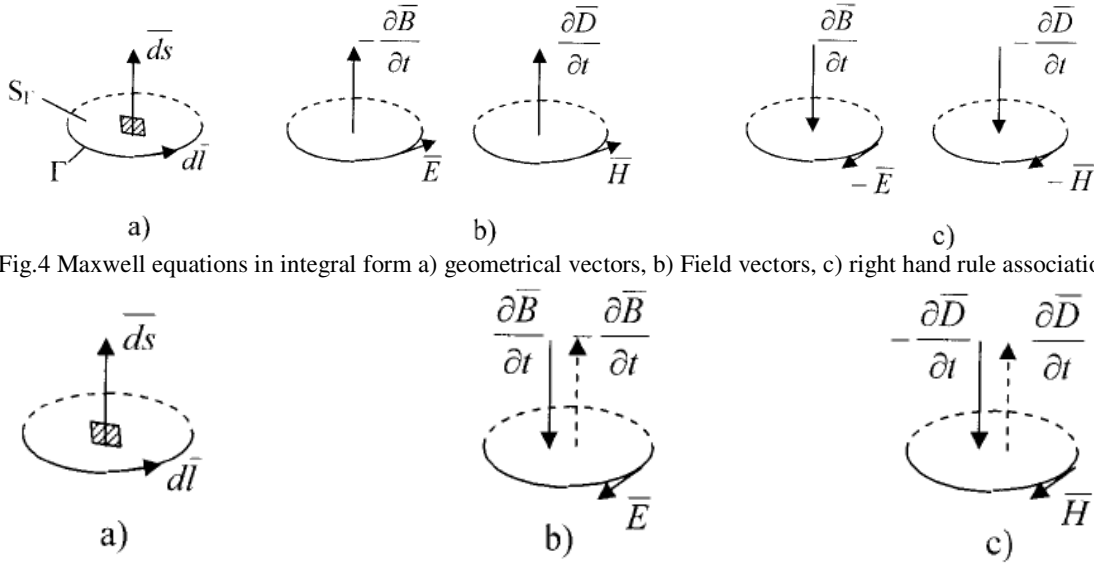


Fig.4 Maxwell equations in integral form a) geometrical vectors, b) Field vectors, c) right hand rule association

Fig.5 Consequences of changing the sign of one term only a)right hand rule for geometrical vector, b) left hand rule for \bar{E} and $\left(-\frac{\partial \bar{E}}{\partial t}\right)$, c) left hand rule for \bar{H} and $\frac{\partial \bar{E}}{\partial t}$.

such that \bar{H} and $\frac{\partial \bar{D}}{\partial t}$ form also a “left hand” system

(Fig.5c). The reciprocal case is true too: changing in eqn.(20) the right side term sign, Faraday’s law (10) comes back to its initial form (15). The interdependence of Faraday’s and Ampere’s laws (equations) reflect the direct physical relationship

between the fields \bar{E} and $\frac{\partial \bar{H}}{\partial t}$ and \bar{H} and $\frac{\partial \bar{E}}{\partial t}$.

This fundamental physical “connection” was, in our opinion, ignored in [1]. Implicitly in [1] the Maxwell equations (3)-(4) and (6)-(7) have been considered independent.

4. Reconsidering the case “ $\varepsilon < 0, \mu < 0$ ”

In this case we appeal to the local form (3’), (4’) of Maxwell equations, correlated with their integral form (15),(16) by Stokes theorem (13).

For $\mu < 0$, the right hand side term sign in (3’) is changed, and thus also the vector fields association rule becomes of “left hand” type; these effects are transmitted implicitly to the right side member of eqn (4’):

$$\nabla \times \bar{E} = |\mu| \frac{\partial \bar{H}}{\partial t}; \quad |\mu| > 0 \quad (21)$$

$$\nabla \times \bar{H} = -\varepsilon \frac{\partial \bar{E}}{\partial t}; \quad \varepsilon > 0 \quad (22)$$

according to eqns (19),(20) and Fig.5b,c.

Considering now, in addition, $\varepsilon < 0$, eqns (21),(22) become

$$\nabla \times \bar{E} = -|\mu| \frac{\partial \bar{H}}{\partial t}; \quad |\mu| > 0 \quad (3'')$$

$$\nabla \times \bar{H} = |\varepsilon| \frac{\partial \bar{E}}{\partial t}; \quad |\varepsilon| > 0 \quad (4'')$$

identical to the initial ones (3’)-(4’), when the vectors have been associated by the “right hand” rule.

However, if simultaneously $\varepsilon < 0$ and $\mu < 0$ as in [1], the signs of right hand side terms in (3’) and (4’) will change also simultaneously such that the equations vectors obtain a “left hand” rule association.

$$\nabla \times \bar{E} = |\mu| \frac{\partial \bar{H}}{\partial t}; \quad (23)$$

$$\nabla \times \bar{H} = -|\varepsilon| \frac{\partial \bar{E}}{\partial t}; \quad (24)$$

It should be noted that this way we have reobtained equations (21)-(22) but valid for $\mu < 0$ and $\varepsilon > 0$!

So we lost a mandatory change in sign. In other words, the materials considered in [1] as “double negative” would be, in reality, only “simple negative”. But then they would have an imaginary refraction index n ”:

$$n'' = \sqrt{\frac{\varepsilon(-\mu)}{\varepsilon_0\mu_0}} = \sqrt{\frac{(-\varepsilon)\mu}{\varepsilon_0\mu_0}} = i \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} = in \quad (25)$$

which is excluded in [1] (quote: “we regard n , ε and μ as real numbers”).

Same observations follow when operating with the plane wave equations (6),(7). For $\mu < 0$ they would become:

$$\bar{k} \times \bar{E} = -\omega |\mu| \bar{H} \quad (26)$$

$$\bar{k} \times \bar{H} = \omega \varepsilon \bar{E}, \quad (27)$$

and if, also, $\varepsilon < 0$:

$$\bar{k} \times \bar{E} = \omega |\mu| \bar{H} \quad (28)$$

$$\bar{k} \times \bar{H} = -\omega |\varepsilon| \bar{E}, \quad (29)$$

(28)-(29) are identical to (6)-(7), where, however, $\mu > 0$ and $\varepsilon > 0$.

5. Reconsidering the “negative refraction index”

If $\bar{E}, \bar{H}, \bar{k}$ vectors would have formed a “left hand” system (as in [1]), described by eqns (9)-(10)-Fig.1b- the refraction index (1) would have become also negative because the wave phase speed, which is also opposed, would be also “negative”: $\bar{v}' = -v\bar{u}_k$; that is:

$$n''' = -\sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} \quad (30)$$

Mathematically this assertion may be possible, because [1]:

$$n^{m2} = \frac{\varepsilon\mu}{\varepsilon_0\mu_0} = n^2 \quad (31)$$

But because eqns (9)-(10) do not describe correctly the hypothetic “meta-materials”, this mathematical possibility (30) has to be ruled out.

6. Conclusion

The assumptions and equations that have led to the concept of “negative refraction”, that is, to the so called “left hand” materials, appear to us unfounded within the classical electrodynamics, reason for which, in our opinion, they should be abandoned. In particular, the negative values of magnetic permeability and electric permittivity can not be introduced simultaneously in Maxwell or plane wave equations, but only successively (the equations are not independent, but interdependent).

Our conclusion is in agreement with the fact that, within the macroscopic electromagnetic theory, electric permittivity and magnetic permeability are strict positive quantities [7].

Even if “double negative” materials ($\varepsilon < 0$, $\mu < 0$) would exist, they would in fact be of “right hand” rule and their equations would be identical for the usual materials ($\varepsilon > 0$, $\mu > 0$). However, in this latter case, eq. (5) would require that the vectors (\vec{D}, \vec{E})

and respectively (\vec{B}, \vec{H}) be opposite, a fact that would entirely modify the classical electrodynamics theory. Assumptions and developments based on microscopic models (semi classical or quantic [8], [9], [10] etc) require a special treatment.

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