MATRIX NODAL EQUATIONS FOR A CIRCUIT
WITH VOLTAGE SOURCES

Firouz Mosharraf, PhD

Professor of Mathematics and Engineering, Department of Mathematics and Science,
Rio Hondo College, 3600 Workman Mill Rd, Whittier, CA 90601
Phone: (562) 463–7560, email: fmosharraf@riohondo.edu

***Abstract**: Writing matrix nodal equations for a circuit that contains no voltage source is simple and straight forward. We expect the presence of voltage sources to simplify nodal analysis; however, the current practice used to write matrix nodal equations when the circuit has voltage sources is rather cumbersome and make the matrices unnecessarily larger. This paper introduces an alternative and simpler approach to the current practice. The paper also shows how to adapt the method for the circuit with dependent sources.*

***Key words**: Floating voltage source, grounded voltage source, matrix equation, modified nodal analysis, network analysis, node voltage, nodal equation, node-voltage equation, supernode*

1. Introduction

We begin by introducing matrix equations and matrix multiplication rule because throughout the paper not only we use this rule to write system of equations in matrix form but also we use it to augment matrix equations or to move terms from one side of the equation to the other side. Consider the following system of n equations and n unknowns.

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = b_n \end{cases} \quad (1)$$

Applying the matrix multiplication rule [1], this system of equations is presented in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ or } \mathbf{Ax} = \mathbf{b} \quad (2)$$

We used red font for the first equation of the system to show how matrix multiplication rule is applied to convert Equation 1 to Equation 2.

WRITING matrix nodal equations by inspection for a circuit that contains no independent voltage source and no dependent source is simple and straight forward [2-5]. The matrix equations for such a circuit with $n + 1$ nodes is

$$\begin{bmatrix} G_{11} & -G_{12} & \cdots & -G_{1n} \\ -G_{21} & G_{22} & & -G_{2n} \\ \vdots & & \ddots & \vdots \\ -G_{n1} & -G_{n2} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} \text{ or } \mathbf{Gv} = \mathbf{i} \quad (3)$$

While the variable (unknown node-voltages) matrix **v** is shared by all rows due to matrix multiplication rule row k of the coefficient matrix **G** and the constant (source) matrix **i** represents Kirchhoff's Current Law (KCL) equation for node k. The positive diagonal G_{kk} of this row is the sum of the conductances directly connected to node k. The negative off diagonal G_{kj} (where $j \neq k$) is the equivalent conductance directly connecting node k to node j. Since $G_{jk} = G_{kj}$, the **G** matrix is symmetrical. The v_k of matrix **v** is unknown node voltage at node k. The i_k of matrix **i** is the sum of all currents entering node k from current sources that are directly connected to node k. As a simple example, consider the circuit of Figure 1. Elements of the coefficient matrix are $G_{11} = 1/3 + 1/1 + 1/5 = 23/15$ S (Siemens), $G_{12} = G_{21} = 1/5$ S and $G_{22} = 1/2 + 1/5 = 7/10$ S. Elements of the current matrix are $i_1 = 3$ A and $i_2 = 8 - 3 = 5$ A. The nodal equation in matrix form for the circuit is

$$\begin{bmatrix} 23/15 & -1/5 \\ -1/5 & 7/10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad (4)$$

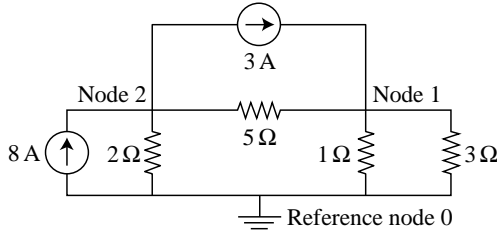Solution of Equation 4 is $v_1 = 3$ V and $v_2 = 8$ V.

Fig. 1- A circuit with no voltage source

In general, the presence of voltage sources simplifies the nodal analysis because it reduces the number of KCL equations. Nonetheless, the current practice of writing matrix nodal equations for the circuits that contain voltage sources is rather cumbersome and makes the matrices unnecessarily larger. One current practice is to use source transformation and replace every voltage source in series with a resistance by an equivalent current source in parallel with the same resistance [3]. Absence of a series resistance makes this method useless. Another current practice is to use *modified nodal analysis* [5].

For example, consider the circuit of Figure 2 with two voltage sources. To facilitate writing nodal equations, resistors in the circuit are expressed in conductance units (Siemens).
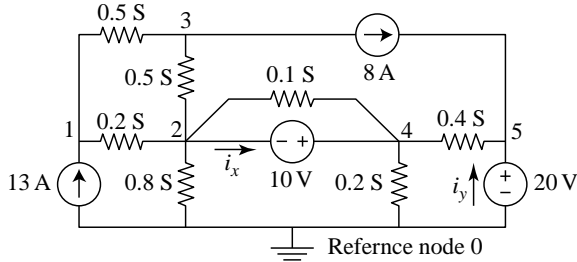

Fig. 2- Circuit with voltage Sources

It is obvious that we cannot use the source transformation method for this circuit because neither voltage source is in series with a resistance. To use modified nodal analysis, we assume the current through the 10-V and 20-V sources are i_x and i_y respectively as shown in the figure and treat i_x and i_y as fictitious current sources. With these assumptions, the matrix nodal equation for the circuit is

$$\begin{bmatrix} 0.7 & -0.2 & -0.5 & 0 & 0 \\ -0.2 & 1.6 & -0.5 & -0.1 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 \\ 0 & -0.1 & 0 & 0.7 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 13 \\ -i_x \\ -8 \\ i_x \\ 8 + i_y \end{bmatrix} \quad (5)$$

This approach introduces two new unknowns (i_x and i_y) and thus two more equations are needed. These two new equations are Kirchhoff's Voltage Law (KVL) equations for the two voltage sources:

$$v_4 - v_2 = 10 \text{ V} \quad (6)$$
$$v_5 = 20 \text{ V} \quad (7)$$

We augment the matrix equation (Equation 5) by two rows to add these two new equations to the matrix:

$$\begin{bmatrix} 0.7 & -0.2 & -0.5 & 0 & 0 \\ -0.2 & 1.6 & -0.5 & -0.1 & 0 \\ -0.5 & 0.5 & 1 & 0 & 0 \\ 0 & -0.1 & 0 & 0.7 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 13 \\ -i_x \\ -8 \\ i_x \\ 8 + i_y \\ 10 \\ 20 \end{bmatrix} \quad (8)$$

Row 6 of the Equation 8 (shaded in color) represents Equation 6. All entries of the **G** matrix of this row is zero except for entries in column two which is −1 (representing $-1v_2$) and in column four which is 1 (representing $1v_4$). Entry in row 6 of the right matrix is 10 which is right side of Equation 6. Similarly, row 7 (shaded in gray) represents Equation 7.

Since unknowns belong to the left side of the equation, we move i_x and i_y terms of Equation 8 to the left and incorporate them into **G** and **v** matrix. We do this by adding two columns to the right of **G** matrix:

$$\begin{bmatrix} 0.7 & -0.2 & -0.5 & 0 & 0 & 0 & 0 \\ -0.2 & 1.6 & -0.5 & -0.1 & 0 & 1 & 0 \\ -0.5 & 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0.7 & -0.4 & -1 & 0 \\ 0 & 0 & 0 & -0.4 & 0.4 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i_x \\ i_y \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ -8 \\ 0 \\ 8 \\ 10 \\ 20 \end{bmatrix} \quad (9)$$

We first add the unknown themselves to row 6 and 7 of the unknown matrix **v**. We then add the opposite of their coefficients to G matrix (opposite because we are moving these terms from one side of the equation to the other side). Transformation follows matrix multiplication rule. Since i_x is added to row 6 of the matrix **v**, coefficients of i_x terms in the **i** matrix of Equation 8 are entered into column 6 of the **G** matrix (shaded in color): Entry 1 in row two of this column represents $-i_x$ in row two of **i** matrix of Equation 8. Entry −1 in row four of this column represents i_x in row four of **i** matrix of Equation 8. Similarly since i_y

is added to row 7 of matrix **v**, coefficients of i_y terms in Equation 8 are entered into column 7 of the **G** matrix (shaded in gray). Solving Equation 9, we get $v_1 = 25$ V, $v_2 = 5$ V, $v_3 = 7$ V, $v_4 = 15$, $v_5 = 20$, $i_x = 2$ A and $i_y = -6$ A.

This paper introduces an alternative and simpler approach to the current practices. The paper also shows how to adapt the method to circuit with dependent sources.

2. Proposed Method

A voltage source in a circuit is either a *grounded voltage source* or it is a *floating voltage source* [3, 5] A grounded voltage source is a voltage source that is connected to the reference node or to a node with a known voltage. A voltage source that is not a grounded voltage source is a floating voltage source. For example, the 20-V source of Figure 2 is a grounded voltage source but the 10-V source in the figure is a floating one. When a node is connected to a grounded voltage source, its voltage is easily determined by inspection or from a simple KVL equation. Nodes that are connected together via floating voltage sources make a supernode [2-6]. We can write $x - 1$ simple KVL equations and one KCL for x nodes that make a supernode.

To write matrix nodal equations for a circuit with $n + 1$ nodes that contains voltage sources, we begin by creating a matrix equation with n rows and make entries for nodes that are not connected to any voltage source as before. Next, we fill in the remaining rows by KVL and KCL equations for the nodes that are connected to voltage sources. To demonstrate this new method and to compare it with the modified nodal analysis we use it to write matrix nodal equations for the circuit of Figure 2 which is shown below with supernode shaded.
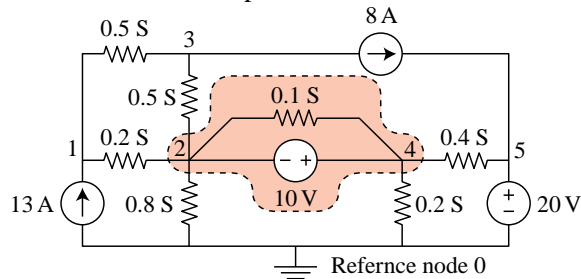

Fig. 3- Circuit with voltage Sources

We begin with a matrix equation with five rows and complete the first and the third rows for nodes 1 and

3 that are not connected to any voltage source. The entries for row 1 and 3 are the same as defined in Equation 3:

$$
\begin{bmatrix}
0.7 & -0.2 & -0.5 & 0 & 0 \\
? & ? & ? & ? & ? \\
-0.5 & -0.5 & 1 & 0 & 0 \\
? & ? & ? & ? & ? \\
? & ? & ? & ? & ?
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{bmatrix}
=
\begin{bmatrix}
13 \\ ? \\ -8 \\ ? \\ ?
\end{bmatrix}
\quad (10)
$$

Node 5 is connected to the grounded 20-V source. The KVL equation for the node is Equation 7 ($v_5 = 20$). Nodes 2 and 4 are connected via a 10-V floating voltage source; they make a supernode (shaded area of Figure 3). The KVL equation for the supernode is Equation 6 ($v_4 - v_2 = 10$). We use two of the remaining rows of the matrix, e.g. row 2 and 4, to represent these two KVL Equations as we did it in Equation 8:

$$
\begin{bmatrix}
0.7 & -0.2 & -0.5 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
-0.5 & -0.5 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
? & ? & ? & ? & ?
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{bmatrix}
=
\begin{bmatrix}
13 \\ 10 \\ -8 \\ 20 \\ ?
\end{bmatrix}
\quad (11)
$$

Finally, we use the last remaining row, row 5, to represent the KCL equation of the supernode. The entries for the KCL equation for the supernode are parallel to entries for other KCL equations. In Equation 3, the sum of the conductances directly connected to node k makes the positive diagonal G_{kk} in the matrix. For the supernode, the sum of the conductances directly connected to each side of the supernode enters the matrix with a positive sign. Since nodes 2 and 4 make the supernode, G_{52} and G_{54} have positive sign in the matrix. They are the sum of conductances outside the supernode that are connected to the supernode at node 2 and at node 4 sides respectively:

$$G_{52} = 0.5 + 0.2 + 0.8 = 1.5 \text{ S} \quad (12)$$
$$G_{54} = 0.4 + 0.2 = 0.6 \text{ S} \quad (13)$$

Note that any conductance inside the supernode (for example 0.1 S) does not appear in the KCL equation of the supernode and thus is not included in the matrix. Each G_{5k} when k is not 2 or 4 is the sum of conductances between the supernode and node k and enters the matrix with a negative sign:

$$G_{51} = 0.2 \text{ S} \quad (14)$$
$$G_{53} = 0.5 \text{ S} \quad (15)$$

$$G_{55} = 0.4 \text{ S} \tag{16}$$

The entry in the right matrix for row four is the sum of all currents from current sources that enter the supernode which for this problem is 0. Entering these values in Equation 11 results in

$$
\begin{bmatrix}
0.7 & -0.2 & -0.5 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
-0.5 & -0.5 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-0.2 & 1.5 & -0.5 & 0.6 & -0.4
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{bmatrix}
=
\begin{bmatrix}
13 \\ 10 \\ -8 \\ 20 \\ 0
\end{bmatrix} \tag{17}
$$

Solution of matrix Equation 17 is $v_1 = 25$ V, $v_2 = 5$ V, $v_3 = 7$ V, $v_4 = 15$ and $v_5 = 20$.

3. Circuits with Dependent Sources

Presence of dependent sources in a circuit results in the constraint equations that express the source current or voltage in terms of some other network currents or voltages. The dependent source current or voltage can ultimately be expressed in term of unknown node voltages. In the matrix form, these terms appear in the **i** matrix but they can be moved to the left side of the equation and incorporated into the **G** and **v** matrices. For example, consider the circuit of Figure 3 where gs are conductances
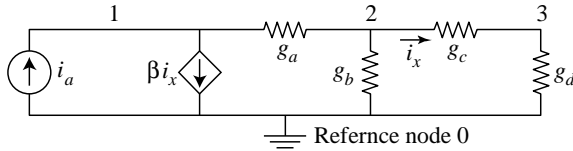

Fig. 4- Circuit with dependent source

The matrix equation for the circuit is

$$
\begin{bmatrix}
g_a & -g_a & 0 \\
-g_a & g_a + g_b + g_c & -g_c \\
0 & -g_c & g_c + g_d
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3
\end{bmatrix}
$$
$$
=
\begin{bmatrix}
i_a - \beta i_x \\ 0 \\ 0
\end{bmatrix} \tag{18}
$$

The constraint equation for the dependent source is

$$i_x = g_c(v_2 - v_3) = g_c v_2 - g_c v_3 \tag{19}$$

Substituting Equation 19 in Equation 18, we get

$$
\begin{bmatrix}
g_a & -g_a & 0 \\
-g_a & g_a + g_b + g_c & -g_c \\
0 & -g_c & g_c + g_d
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3
\end{bmatrix}
$$
$$
=
\begin{bmatrix}
i_a - \beta g_c v_2 + \beta g_c v_3 \\ 0 \\ 0
\end{bmatrix} \tag{20}
$$

We need to move $-\beta g_c v_2$ and $\beta g_c v_3$ terms in the first row of the **i** matrix to the left side of the equation because they contain unknown node voltages. Since we are moving these terms from one side of the equation to the other side, we first change their signs (to $\beta g_c v_2$ and $-\beta g_c v_3$). We use matrix multiplication rule backward to incorporate $\beta g_c v_2$ term into the **G** matrix: We drop v_2 from the term and add βg_c to G_{12}. We drop v_2 because every term of G_{12} is multiplied by v_2. Similarly, to move $-\beta g_c v_3$ term into **G** matrix we drop v_3 from the term and add $-\beta g_c$ to G_{13}. The resulting matrix is

$$
\begin{bmatrix}
g_a & -g_a + \beta g_c & -\beta g_c \\
-g_a & g_a + g_b + g_c & -g_c \\
0 & -g_c & g_c + g_d
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3
\end{bmatrix}
=
\begin{bmatrix}
i_a \\ 0 \\ 0
\end{bmatrix} \tag{21}
$$

By multiplying **G** matrix by **v** matrix and setting it to the **i** matrix, we can verify that Equations 21 is equivalent to Equation 20.

4. Conclusion

One of the two current methods, using source transformation method, for writing matrix nodal equations for a circuit with voltage sources is often useless. The second method, modified nodal analysis, is rather cumbersome and makes the matrices unnecessarily larger. A simpler alternative method is to fill in the rows of the matrix for nodes that are not connected to any voltage source as usual and then complete the matrix by using the remaining rows to represent KVL and KCL equations associated with nodes that are connected to voltage sources. The method can easily be adapted for circuits that contain dependent sources.

5. References

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