

MINIMUM TIME CONTROL OF A MOBILE ROBOT

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Abstract: In this article we present a simple algorithm that finds the minimum time path for a two-wheeled vehicle. The proposed algorithm takes into account constraints on angular velocities and accelerations, thus, enabling smooth optimal trajectories. The dynamical case is studied, as well. Simulations and results are presented.

Key words: mobile robot, minimum time control, constrained optimization.

1. Introduction

A major motivation using mobile robots is to increase productivity, since robots are often slower than what we expect. The present paper deals with minimum time trajectory generation for a moving vehicle from an initial configuration to a final strategy with free final orientation. Obviously, minimizing time process is equivalent to maximizing robot's velocity and therefore optimizing productivity.

The special kind of mobile robots of our interest is differential drive with bounded velocity, and bounded acceleration.

Two major goals are to be achieved, first to write the kinetic system as a set of difference equations, second to develop a control law that enables path following optimization.

In [4] only straight line and turning in place motions were generated, regardless of constraints that might exist during control process.

[5] and [6] however, considered steered vehicles which is a restricted case.

2. Problem formulation

Let P a point with coordinates (x, y) in the fixed frame (o, x, y) , and coordinates (x_M, y_M) in the mobile frame (o', x', y') . Consider also (x_0, y_0)

coordinates of the mobile frame's origin with respect to the fixed frame (Fig. 1). The kinetic system is written in terms of the following system equations:

$$\begin{aligned} \dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \psi \end{aligned} \quad (1)$$

Such that: $v = \frac{r}{2}(w_r + w_l)$ and $\psi = \frac{r}{2L}(w_r - w_l)$

Where w_r and w_l are the right and left angular wheel speeds respectively.

It is easy to show the following:

$$x = x' \cos \theta - y' \sin \theta + x_0 \quad (2)$$

$$y = x' \sin \theta + y' \cos \theta + y_0 \quad (3)$$

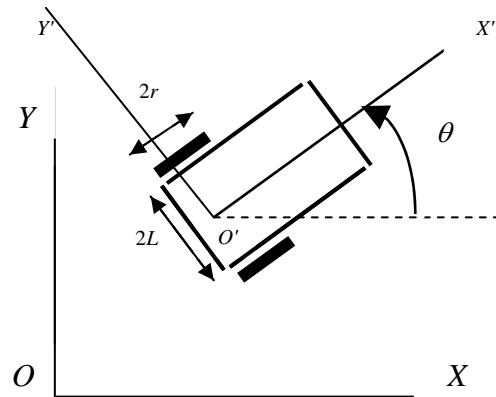


Fig1. System kinematics

Now suppose a displacement of the point P as shown on figure 2. We can, then, express the curvature radius R as.

$$R = L \frac{w_r + w_l}{w_r - w_l} \quad (4)$$

In addition, we have:

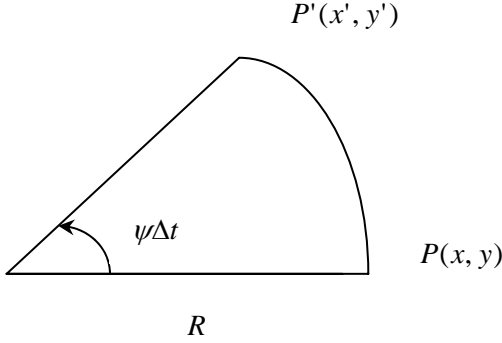


Fig2. Small angle variation

$$\begin{aligned} x' &= R \sin(\Delta\theta) \\ y' &= R(1 - \cos(\Delta\theta)) \end{aligned}$$

Using equations (2), (3) we get:

$$x = \frac{v}{\psi} \{ \cos \theta \sin(\psi\Delta t) - (1 - \cos(\psi\Delta t)) \sin \theta \} + x_0 \quad (5)$$

$$y = \frac{v}{\psi} \{ \sin \theta \sin(\psi\Delta t) + (1 - \cos(\psi\Delta t)) \cos \theta \} + y_0 \quad (6)$$

$$\theta = \psi\Delta t + \theta_0 \quad (7)$$

The goal is to solve system (1) (same as equations (5)-(7)) such that the following cost:

$$J = \int_{t_0}^{t_f} dt$$

is minimised, under the constraints:

$$\begin{aligned} |w_r| &\leq w_{rMAX} \\ |w_l| &\leq w_{lMAX} \\ |\psi| &\leq \psi_{MAX} \\ \left| \frac{d\psi}{dt} \right| &\leq \tau_{MAX} \end{aligned}$$

3. Pontryagin's maximum principle

The hamiltonian is:

$$H = 1 + \lambda_1 v \cos \theta + \lambda_2 v \sin \theta + \lambda_3 \psi \quad (8)$$

Where λ_i $i=1,2,3$ are Lagrange multipliers, which must satisfy the following:

$$\frac{\partial H}{\partial x} = -\dot{\lambda}_1 = 0; \quad \frac{\partial H}{\partial y} = -\dot{\lambda}_2 = 0; \quad (9)$$

$$\frac{\partial H}{\partial \theta} = -\dot{\lambda}_3 = -\lambda_1 v \sin \theta + \lambda_2 v \cos \theta \quad (10)$$

We must also have:

$$H(t_f) = 0 \quad (11)$$

And the Hamiltonian must be minimal along an optimal path $\forall t \in [t_0, t_f]$.

Set of equations (9) give trivial solutions:

λ_1 and λ_2 constants, however,

$$\lambda_3 = \int_{t_0}^{t_f} v(\lambda_1 \sin \theta - \lambda_2 \cos \theta) dt$$

One can study each case of the four possibilities:

- 1: $w_r = \pm w_{rMAX}$ $w_l = \pm w_{lMAX}$
- 2: $w_r = \pm w_{rMAX}$ $|w_l| \leq w_{lMAX}$
- 3: $|w_r| \leq w_{rMAX}$ $w_l = \pm w_{lMAX}$
- 4: $|w_r| \leq w_{rMAX}$ $|w_l| \leq w_{lMAX}$

The first case is a degenerate one, in which the robot either follows a straight line or turns in place.

4. Path following

Intuitively, the robot should be heading towards the target in order to guarantee optimality, if we let:

$\Delta\theta(t) = \theta_T(t) - \theta(t)$ be the instantaneous error angle, i.e., difference between target angle and actual angle,

whereas, $\theta_T(t) = \arctg\left(\frac{\Delta y}{\Delta x}\right)$ such that

$\Delta y = y_T - y_i$ and $\Delta x = x_T - x_i$, (x_i, y_i) are actual coordinates of the center of mass of the robot and (x_T, y_T) are target coordinates.

Therefore, one can propose the following control:

$$\psi = K_P \frac{\Delta\theta}{\Delta t} + K_I \theta \quad (13)$$

K_P and K_I are regulating gains.

5. Robot dynamics

Let τ_r and τ_l be the right and left torques applied on the right and left wheels respectively, therefore, the equations of motion are:

$$m\dot{v} = -F_v v + (\tau_r + \tau_l)/r \quad (14-1)$$

$$J\dot{\psi} = -F_\psi \psi + L(\tau_r - \tau_l)/r \quad (14-2)$$

Where m is the mass of the robot, J the moment of inertia, F_v and F_ψ are the viscous and rotational friction coefficients respectively.

Now, let us consider the following state vector:

$$\mathbf{x} = (x \quad y \quad \theta \quad v \quad \psi)^T$$

And the input vector: $\mathbf{u} = (\tau_r \quad \tau_l)^T$

In such a way that:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} \quad (15-1)$$

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & \sin \theta & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -F_v/m & -F_\psi/J \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (15-2)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/rm & 1/rm \\ L/rJ & -L/rJ \end{pmatrix} \quad (15-3)$$

On a given path trajectory s , by using the curvature radius $R(s)$ and combining equations (14-1) and (14-2) we get:

$$\dot{s} = v \quad (16-1)$$

$$\dot{v} = a_0(s, v) + a_1(s)\tau_r \quad (16-2)$$

$$b(s) = R(s)(J + mL R(s))$$

$$a_0(s, v) = \frac{JR'(s)v^2 - R(s)(F_\psi + F_vLR(s))v}{b(s)}$$

$$a_1(s) = \frac{2LR^2(s)}{rb(s)}$$

6. Motion generation

The aim goal of this section is to design the minimum time trajectory under torque limitations. We can, therefore, state the following problem:

$$\text{Minimize } \int_0^{t_f} dt$$

Under the constraints:

System (16),

$$\tau_r \leq \tau_m \quad \tau_l \leq \tau_m \quad (17)$$

Where assuming identical wheel characteristics.

According to [7], constraints on the applied torques can be transformed into constraints on acceleration in the phase plane (s, v) such that:

$$\ddot{s}_{min} \leq \ddot{s} \leq \ddot{s}_{max}$$

Where:

$$\ddot{s}_{min} = \begin{cases} -a_1(s)\tau_m + a_0(s, v) & \text{if } a_1(s) > 0 \\ a_1(s)\tau_m + a_0(s, v) & \text{if } a_1(s) < 0 \end{cases}$$

$$\ddot{s}_{max} = \begin{cases} a_1(s)\tau_m + a_0(s, v) & \text{if } a_1(s) > 0 \\ -a_1(s)\tau_m + a_0(s, v) & \text{if } a_1(s) < 0 \end{cases}$$

And the maximum velocity occurs when:

$$\ddot{s}_{min} = \ddot{s}_{max}$$

Once the maximum velocity is found, forward and backward integration of system (16) is performed until matching is found, exactly, that will be the switching point. We should notice that, we are dealing with a non-singular case, that is, the resulting optimal controls are of bang-bang type.

In the left hand side of equation (16-2), while taking the discrete time version, the following approximation is considered:

$$v_{k+1}^2 - v_k^2 / 2\Delta; \quad \Delta = s_{k+1} - s_k$$

For instance, if we neglect viscous and friction coefficients we get the following recursive formula:

$$v_{k+1}^2 = \frac{\Delta R^2(s)}{b(s)} \left\{ \frac{\pm 4L\tau_m}{r} + \frac{b(s) + 2\Delta JR'(s)}{\Delta R^2(s)} v_k^2 \right\}$$

7. Simulations

First, we study the kinetic case:

Example 1: In this case, no constraint is considered. The wheel radius is taken $r = .1m$ and the linear velocity is supposed to be constant such that $v = 1m/s$ initial position is defined as $x_0 = 0$ and $y_0 = 0$ $\theta_0 = 0$ meanwhile the final position is $x_f = 1m$ and $y_f = 1m$ with free final orientation.

Gain constants are chosen $K_P = .2$ and $K_I = .2$, the stepwise is $\Delta t = .05s$.

The final time is found to be $t_f = 1.35s$, the resulting trajectory is shown in figure 3.

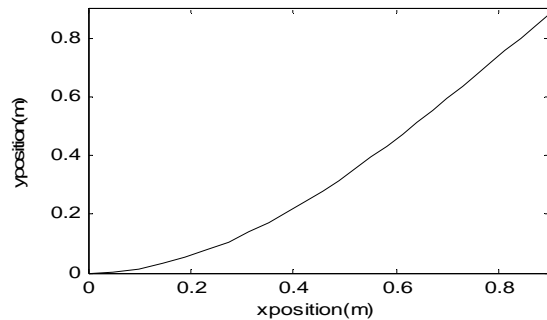


Fig.3. Phase plane with zero initial angle

At the same time the angular position and velocity are shown in figures 4 and 5.

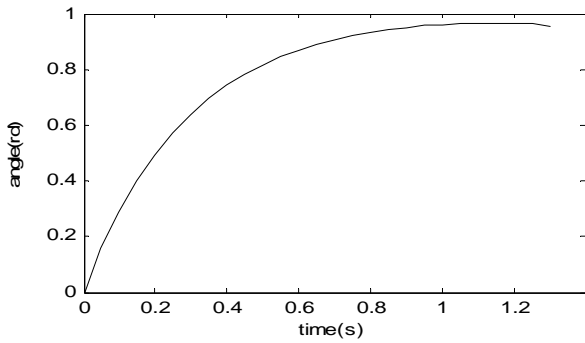


Fig.4. Robot orientation as a function of time.

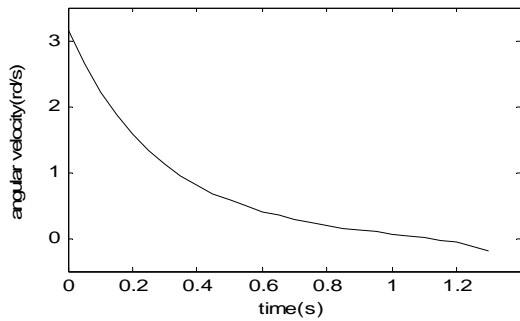


Fig.5. Robot's angular velocity.

Right and left wheel's angular velocities are plotted in figures 6 and 7 respectively.

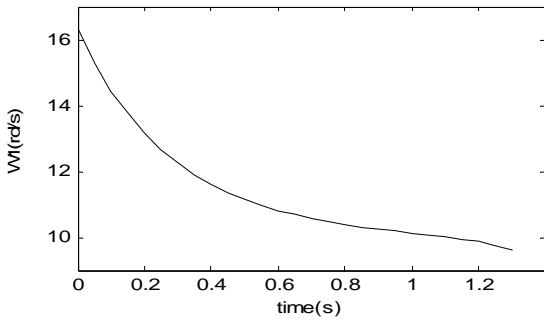


Fig. 6. Right wheel angular velocity.

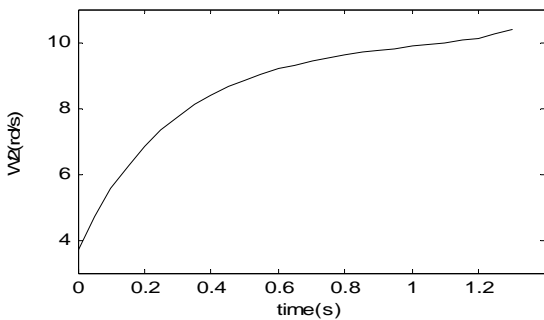


Fig. 7. Left wheel angular velocity.

Example 2: The same last values are considered again, additionally we take $\psi_{MAX} = 1rd / s$.

Consequently, the final time is found to be $t_f = 1.5s$, and the resulting phase trajectory is sketched in figure 8.

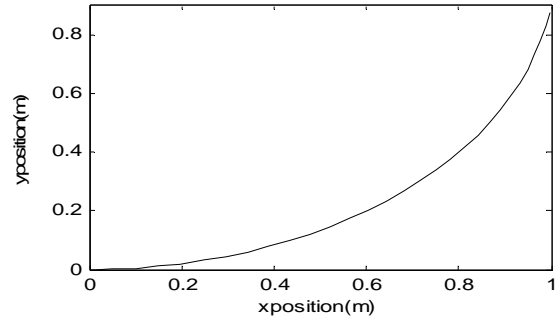


Fig.8. Constrained trajectory.

In addition, right and left wheel velocities are presented in figures 9 and 10 respectively.

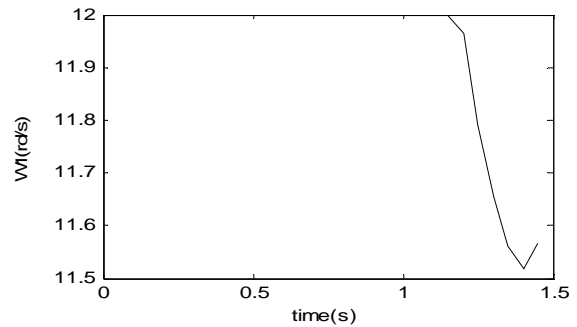


Fig.9. Right wheel velocity.

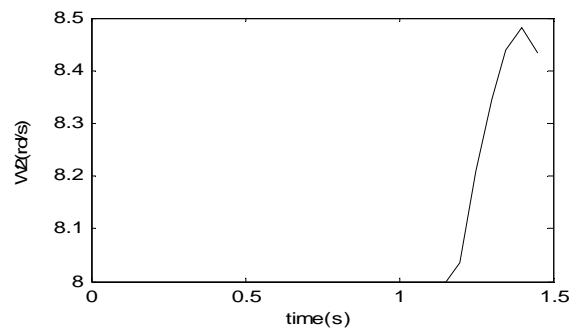


Fig. 10. Left wheel velocity.

The cart experimentally used to test the mobile robot performances, and which the physical characteristics are employed in the following simulations is a two wheeled differentially driven robot with a free wheel (caster). Each driven wheel is connected to a DC motor controlled by a pulse width modulation servo-

controller. The whole system is powered with a simple 2X12V set of batteries. The sampling rate is 20ms. In the following we consider the dynamics of the robot with the following data:

$$m = 27 \text{ kg}, \quad J = 1.43 \text{ Kg. m},$$

$$L = .23\text{m}, \quad \tau_m = 5.9 \text{ N. m},$$

Example 3: Let the path be a unit circle, which means that $R(s) = 1$, by applying the results found in the last section we find that the switching point occurs at $s = 2m$ as shown in figure 11.

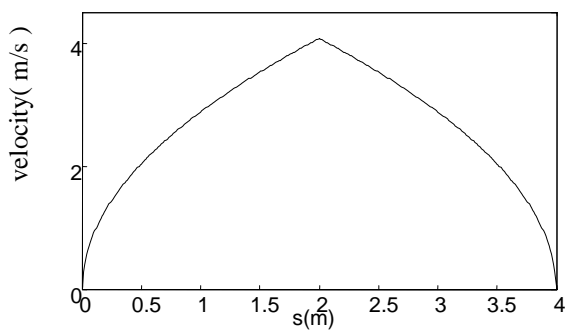


Fig.11. Velocity profile.

We can also show the optimal torque τ_r variations in figure 12.

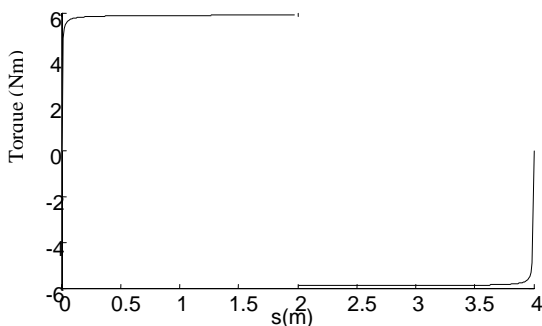


Fig.12.Optimal right torque.

Where, the trajectory variations are sketched in figure 13.

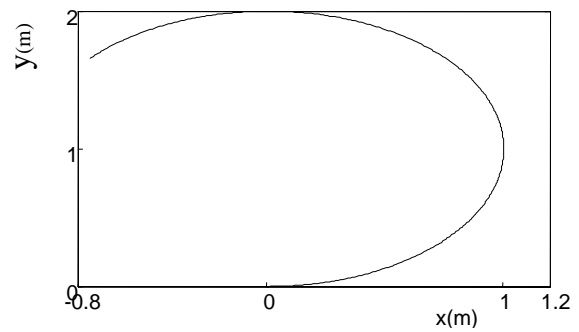


Fig. 13. Position trajectory

We can also show the path trajectory with respect to time.

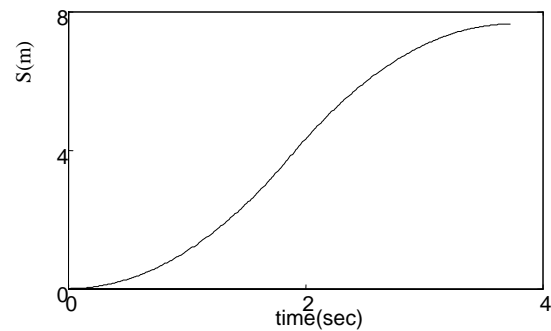


Fig. 14. Time variation of path trajectory.

8. Conclusions

We developed a simple algorithm that finds minimum time trajectory for a differential drive robot taking into account constraints on angular velocity and/or acceleration. The actual algorithm may be extended to the general dynamic case, which will be the main scope of our future research, besides, the possibility of static obstacles existing in work space of the robot.

Another future perspective would be the situations, in which final orientation of the robot is specified.

For practical reasons, one could use the resulting optimal solutions as model references to real robot plant, in such a way that real time optimal path following can be achieved.

References

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