

TRANSLATION DYNAMICAL NONLINEAR MODEL IN PERTURBED KEPLERIAN CONDITIONS FOR A GEOSTATIONARY SATELLITE

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Abstract--The dynamics of a GEO satellite will be studied in this work to obtain a dynamical model as accurate as possible. This model will be obtained in terms of Gauss' variation of osculating parameter (VOP) equations containing the environmental perturbing accelerations, which are traditionally used to plan the station keeping maneuvers. The idea is to implement a controller for geostationary station keeping purposes based on a model written in terms of osculating orbital elements instead of averaged elements. Such a controller plans in an automatic way the station keeping (SK) maneuvers and it could be integrated on board in view of autonomous station keeping control loop.

Index Terms--classical orbital elements COE; equinoctial orbital elements EOE; environmental forces acting; perturbing accelerations; variation of parameter (VOP).

1. INTRODUCTION

The Geostationary Earth Orbit (GEO) satellites maintain an essentially fixed position with respect to the surface of the Earth. This is made possible by inserting the spacecraft into a circular, equatorial orbit at an altitude of roughly 36000 km. At this altitude, if the main environmental disturbing forces (the Earth's non-spherical gravity attraction, the Moon's and Sun's gravity attraction and the solar radiation pressure) are neglected except the Kepler attraction of the Earth, the geostationary motion is ideal. The satellite remains fixed with respect to the surface of the Earth. The mean motion n matches the Earth's rotation rate ω_{\oplus} of one revolution per 23 hours and 56 minutes.

In presence of perturbations, it is a common practice to control a GEO satellite actively via station keeping maneuvers such that it stays confined in a box of 100–150 km width around a nominal geostationary longitude and latitude [1]. Traditionally this is done with an open loop control technique based on a dynamical model of the satellite state vector subject to the only environmental perturbing forces and on a separate dynamical model taking into account the only thrust effects. Moreover, this latter model is derived supposing to use a chemical propulsion system characterized by high thrusts and very short thrust durations relative to the orbital period. These impulsive thrust hypotheses lead to assume that maneuvers induce jumps in the velocity part of the state vector but not in the position part. The GEO station keeping problem is thus dealt with in a discrete way, without considering spacecraft acceleration but only velocity and position vectors.

With a view to substitute chemical propulsion systems with electrical ones, the solution approaches of the GEO station keeping problem should become continuous, in order to gain benefit from the technology change. A GEO satellite dynamics model has to be obtained taking into account all the perturbing forces (environmental or not) acting on the spacecraft.

In this paper, we explain in detail the nonlinear dynamical model used to design the station keeping controller and the analytical approximations done to implement this model. We present validation simulation results, which have been obtained implementing Gauss' VOP equations [2].

2. NONLINEAR MODEL IN PERTURBED KEPLERIAN CONDITIONS

When the satellite is subject to disturbing forces different to the Keplerian Earth's gravity attraction, the above properties of orbital elements are no longer true. Under the hypothesis that the disturbing forces are weak with respect to the main Keplerian two-body term, it is reasonable to think that the satellite trajectory is slightly different from a conic, i.e., that the six first integrals of the unperturbed two-body equations undergo only weak variations. The osculating elliptic trajectory defined by the osculating elements is tangent to the actual trajectory (i.e., it has the same velocity vector) but it doesn't have the same curvature radius (i.e., it has a different acceleration vector). According to the variation of parameter (VOP) methods originally developed by Euler and later improved by Lagrange and Gauss in the 18th century to analyses the perturbation effects [2], the vectorial first order differential dynamical equations valid for a GEO satellite subject to the Keplerian Earth's gravity attraction and to the generic not Keplerian disturbing acceleration a_d are

$$\frac{dr}{dt} = v \quad (1)$$

$$\frac{dv}{dt} = -GM_{\oplus} \frac{r}{\|r\|^3} + a_d(r, v, t) \quad (2)$$

Where r the spacecraft position vector, v is the spacecraft velocity vector, GM_{\oplus} is the gravitational coefficient of the earth.

And they are equivalent to the set of differential equations

$$\frac{dx_{EOE}}{dt} = K(x_{EOE}) - \omega_{\oplus} + D(x_{EOE}, t) \quad (3)$$

With

$$x_{EOE} = [a \ P_1 \ P_2 \ Q_1 \ Q_2 \ l_\theta]^T \quad (4)$$

$$\omega_\oplus = [0 \ 0 \ 0 \ 0 \ 0 \ \omega_\oplus]^T \quad (5)$$

$$K = [0 \ 0 \ 0 \ 0 \ 0 \ n]^T \quad (6)$$

Where a is the semi-major axis, (P_1, P_2) is the eccentricity, (Q_1, Q_2) is the Inclination and l_θ is the mean longitude in terms equinoctial orbital elements EOE's [3].

The equinoctial element set is sometimes expressed in a slightly different fashion. Elements Q_1 and Q_2 can be defined with $\sin(i/2)$. The elements Q_1 and Q_2 to represent the projection of the vector oriented in the direction of the ascending node with a different magnitude onto the Q and E directions, respectively.

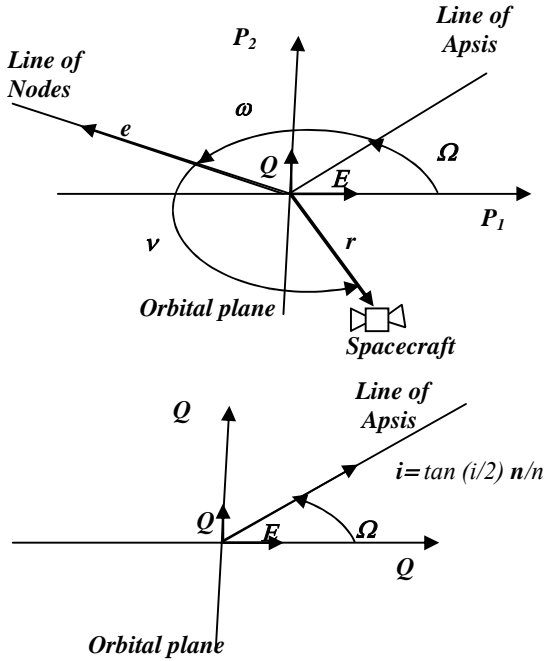


Fig.1. Equinoctial orbital elements (EOEs)

Where equinoctial orbital elements (EOEs) are

$$P_1 = e \sin(\omega + \Omega) \quad (7)$$

$$P_2 = e \cos(\omega + \Omega) \quad (8)$$

$$Q_1 = \tan(i/2) \sin(\Omega) \quad (9)$$

$$Q_2 = \tan(i/2) \cos(\Omega) \quad (10)$$

$$l_\theta = \Omega + \omega + M - \Theta(t) \quad (11)$$

Where $a, e, i, \Omega, \omega, M$ are the semi-major axis, eccentricity, Inclination, mean longitude, Argument of perigee and mean anomaly in the classical orbital elements (COEs) [3], respectively. Also called the Keplerian elements, $\Theta(t)$ is Right ascension of the Greenwich meridian, ω_\oplus is the angular velocity of the Earth.

Where the mean motion n is function of the unperturbed semi-major axis a .

$$n = \sqrt{\frac{GM_\oplus}{a^3}} \quad (12)$$

Vector D is the disturbing contribution to the VOP equations, small with respect to the Keplerian part [3]. We will call this perturbing contribution G (as Gauss) when the VOP equations contain directly the disturbing acceleration a_d

$$D(x_{EOE}, t) \equiv G(x_{EOE}, t) \propto a_d(x_{EOE}, t) \quad (13)$$

3. GAUSS VARIATION OF PARAMETER (VOP) EQUATION

The Gauss variation of parameter equations (or Gauss planetary equations) give the variations over time of the classical orbit elements COEs characterizing the motion of a spacecraft subject to the Keplerian gravity attraction of the Earth's and subject to small perturbing accelerations [4].

Let be

$$a_d = a_{dR}R + a_{dT}T + a_{dN}N \quad (14)$$

The sum of all the disturbing acceleration vectors expressed in the radial tangent normal RTN reference frame.

The disturbing acceleration components perturb the solution x_{KCOE} of the unperturbed Kepler's problem [5]. This new perturbed solution fulfils the Gauss' variation of parameter equations

$$\frac{dx_{COE}}{dt} = K(x_{COE}) + \tilde{G}(x_{COE}) a_d(x_{COE}, t) \quad (15)$$

Where

$$x_{COE} = [a \ e \ i \ \Omega \ \omega \ M]^T \quad (16)$$

$$\tilde{G} = \begin{bmatrix} \frac{2e \sin v}{n\sqrt{1-e^2}} & \frac{2(1+e \cos v)}{n\sqrt{1-e^2}} & 0 \\ \frac{\sqrt{1-e^2} \sin v}{na} & \frac{\sqrt{1-e^2} (\cos E + \cos v)}{na} & 0 \\ 0 & 0 & \frac{r \cos(\omega+v)}{na^2 \sqrt{1-e^2}} \\ 0 & 0 & \frac{r \sin(\omega+v)}{na^2 \sqrt{1-e^2}} \\ -\frac{\sqrt{1-e^2} \cos v}{nae} & \frac{\sqrt{1-e^2}}{nae} (1 + \frac{1}{1+e \cos v}) \sin v & -\frac{r \cos i}{na^2 \sqrt{1-e^2}} \frac{\sin(\omega+v)}{\sin i} \\ \frac{1-e^2}{nae} (\cos v - \frac{2e}{1+e \cos v}) & -\frac{1-e^2}{nae} (1 + \frac{1}{1+e \cos v}) \sin v & 0 \end{bmatrix} \quad (17)$$

The dependence of the matrix \tilde{G} on the mean anomaly M is not explicit. In fact, sine and cosine of the true anomaly v depend on sine and cosine of the eccentric anomaly E [5], via the inverse formulas

$$\sin v = \frac{\sin E \sqrt{1-e^2}}{1-e \cos E} \text{ and } \cos v = \frac{\sin E - e}{1-e \cos E} \quad (18)$$

In its turn, the eccentric anomaly E is solution of Kepler's equation

$$E - e \sin E = M \quad (19)$$

This, for small eccentricity, can be solved analytically with a Taylor series expansion process or with a differential process [6]. It can also be easily solved numerically, using the method of successive substitutions or using Newton's method to calculate successive refinements of E values until the result changes by less

than a specified amount from one iteration to the next. In this last solution way, an auxiliary function $f(E) = E - e \sin E - M$ (20) Is defined and solved for a given value of M . Applying Newton's method for this purpose, an approximate root E_i of f may be improved by computing:

$$E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)} \quad (21)$$

Gauss' VOP equations (15) can also be written in terms of the equinoctial orbit elements (EOE) thanks to the following differential conversion formula [7]:

$$\frac{dx_{EOE}}{dt} = \nabla_{COE}(x_{EOE}) \frac{dx_{COE}}{dt} - \omega_{\oplus} \quad (22)$$

Where Jacobian matrix [8]

$$\nabla_{COE} x_{EOE} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P_1}{\sqrt{P_1^2 + P_2^2}} & 0 & P_2 & P_2 & 0 \\ 0 & \frac{P_2}{\sqrt{P_1^2 + P_2^2}} & 0 & -P_1 & -P_1 & 0 \\ 0 & 0 & \frac{Q_1(1+Q_1^2+Q_2^2)}{2\sqrt{Q_1^2+Q_2^2}} & Q_2 & 0 & 0 \\ 0 & 0 & \frac{Q_2(1+Q_1^2+Q_2^2)}{2\sqrt{Q_1^2+Q_2^2}} & -Q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (23)$$

Is the partial derivative matrix of the equinoctial orbit element state vector x with respect to the classical orbit element state vector x_{COE} . The right-hand side of (15) has to be replaced in the conversion formula (22) by matrix \tilde{G} and disturbing acceleration vector a_d expressed in terms of equinoctial elements

$$\tilde{G}(x_{COE}) \rightarrow \tilde{G}(x_{EOE}, t) \quad (24)$$

$$a_d(x_{COE}) \rightarrow a_d(x_{EOE}, t) \quad (25)$$

Gauss VOP equations in terms of equinoctial orbital elements EOE are as follows [9]

$$\frac{dx_{EOE}}{dt} = K(x_{EOE}) - \omega_{\oplus} + G(x_{EOE}, t) \quad (26)$$

Where

$$G(x_{EOE}, t) = g(x_{EOE}, t) a_d(x_{EOE}, t) \quad (27)$$

With

$$g(x_{EOE}, t) = (\nabla_{COE} x_{EOE}) \tilde{G}(x_{EOE}, t) \quad (28)$$

More precisely

$$\nabla_{COE} x_{EOE} = \begin{bmatrix} \frac{2a^2}{h} X_1 & \frac{2a^2 p}{h r} & 0 \\ -\frac{p}{h} \cos L & \frac{r}{h} (P_1 + (1 + \frac{P_2}{r}) \sin L) & -\frac{r}{h} P_2 Y_1 \\ \frac{p}{h} \sin L & \frac{r}{h} (P_2 + (1 + \frac{P_1}{r}) \cos L) & \frac{r}{h} P_1 Y_1 \\ 0 & 0 & \frac{r}{2h} P_2 Y_2 \sin L \\ 0 & 0 & \frac{r}{2h} P_2 Y_2 \cos L \\ -\frac{r}{h} (\frac{a}{a+b} (\frac{P_1}{r} X_2 + \frac{2b}{a})) & \frac{r}{h} (\frac{a}{a+b} (1 + \frac{P_1}{r}) X_1) & \frac{r}{h} Y_1 \end{bmatrix} \quad (29)$$

Where

$$b = a \sqrt{1 - P_1^2 - P_2^2} \quad (30)$$

$$h = nab \quad (31)$$

$$\frac{p}{r} = 1 + P_1 \sin L + P_2 \cos L \quad (32)$$

$$\frac{r}{h} = \frac{h}{GM_{\oplus} (1 + P_1 \sin L + P_2 \cos L)} \quad (33)$$

$$X_1 = P_2 \sin L - P_1 \cos L \quad (34)$$

$$X_2 = P_1 \sin L - P_2 \cos L \quad (35)$$

$$Y_1 = Q_1 \cos L - Q_2 \sin L \quad (36)$$

$$Y_2 = 1 + Q_1^2 + Q_2^2 \quad (37)$$

Sin and cosine of the true longitude L are defined by

$$\sin L = \frac{(\sqrt{1-P_1^2-P_2^2}+1-P_2^2) \sin K + P_1 P_2 \cos K - P_1(1+\sqrt{1-P_1^2-P_2^2})}{(1-P_1 \sin K - P_2 \cos K)(1+\sqrt{1-P_1^2-P_2^2})} \quad (38)$$

$$\cos L = \frac{(\sqrt{1-P_1^2-P_2^2}+1-P_1^2) \cos K + P_1 P_2 \sin K - P_2(1+\sqrt{1-P_1^2-P_2^2})}{(1-P_1 \sin K - P_2 \cos K)(1+\sqrt{1-P_1^2-P_2^2})} \quad (39)$$

Witch are implicit function of $(l_{\theta} + \Theta)$ via the eccentric longitude

$$k = (l_{\theta} + \Theta) - P_1 \cos(l_{\theta} + \Theta) + P_2 \sin(l_{\theta} + \Theta) \quad (40)$$

First order analytical solution in P_1 and P_2 of Kepler's equation

$$f(k; P_1, P_2) = k + P_1 \cos k - P_2 \sin k - (l_{\theta} + \Theta) = 0 \quad (41)$$

Considering P_1 and P_2 as small parameters, the solution of Kepler's equation above has the functional form $k^* = g(P_1, P_2)$ that can be expanded in Taylor series about $P_1=0$ and $P_2=0$

$$k^* = k_0 + k_{11} P_1 + k_{12} P_2 + k_{21} P_1^2 + k_{22} P_2^2 + k_{23} P_1 P_2 + \dots \quad (42)$$

Where k_0 is called the zeroth-order solution, $k_0 + k_{11} P_1 + k_{12} P_2$ is called the first-order solution and so on.

Equation (42) can be rewritten as $k^* = k_0 + \Delta k$ with

$$\Delta k = k_{11} P_1 + k_{12} P_2 + k_{21} P_1^2 + k_{22} P_2^2 + k_{23} P_1 P_2 + \dots \quad (43)$$

Then, k^* is substituted into equation (42) which in turn is expended in terms of small quantities Δk , P_1 and P_2 around $\Delta k = k_0$, $P_1=0$ and $P_2=0$

$$\begin{aligned} f(k; P_1, P_2) &= f(k_0 + \Delta k; P_1, P_2) \\ &= f(k_0; 0, 0) + f_k(k_0; 0, 0) \Delta k + f_{P_1}(k_0; 0, 0) P_1 + f_{P_2}(k_0; 0, 0) P_2 + \\ &+ f_{P_1 k}(k_0; 0, 0) P_1 \Delta k + f_{P_2 k}(k_0; 0, 0) P_2 \Delta k + f_{P_1 P_2}(k_0; 0, 0) P_1 P_2 + \\ &+ \frac{1}{2!} [f_{kk}(k_0; 0, 0) \Delta k^2 + f_{P_1 P_1}(k_0; 0, 0) P_1^2 + f_{P_2 P_2}(k_0; 0, 0) P_2^2] + \\ &+ \dots = 0 \end{aligned} \quad (44)$$

Finally, equation (44) is written only in terms of powers of P_1 and P_2 thanks to equation (43), and the coefficients of the various powers are equated to zero. This process gives the equations

$$f(k_0;0,0) = 0 \quad (45)$$

$$f_k k_{11} + f_{P_1} = 0 \quad (46)$$

$$f_k k_{12} + f_{P_2} = 0 \quad (47)$$

$$f_k k_{21} + f_{P_1 k} k_{11} + \frac{1}{2!} f_{kk} k_{11}^2 + \frac{1}{2!} f_{P_1 P_1} = 0 \quad (48)$$

$$f_k k_{22} + f_{P_2 k} k_{12} + \frac{1}{2!} f_{kk} k_{12}^2 + \frac{1}{2!} f_{P_2 P_2} = 0 \quad (49)$$

$$f_k k_{23} + f_{P_2 k} k_{11} + f_{P_1 k} k_{12} + f_{kk} k_{11} k_{12} + f_{P_1 P_2} = 0$$

⋮

(50)

This can be solved sequentially for k_0 , k_{11} , k_{12} , etc.. All the partial derivatives in the above equations (45)-(50) are evaluated in $k=k_0$, $P_1=0$ and $P_2=0$. At the first order in P_1 and P_2 one obtains the coefficients

$$k_0 = l_\theta + \Theta \quad (51)$$

$$k_{11} = -\frac{f_{P_1}(k_0;0,0)}{f_k(k_0;0,0)} = -\frac{\cos k_0}{1} = -\cos(l_\theta + \Theta) \quad (52)$$

$$k_{12} = \frac{f_{P_2}(k_0;0,0)}{f_k(k_0;0,0)} = \frac{\sin k_0}{1} = \sin(l_\theta + \Theta) \quad (53)$$

Which are those of solution of equation (40). With those *EOE* time histories obtained above, we have evaluated numerically the components F_{dR} , F_{dT} , F_{dN} and the modulus F_d of the vector of the environmental forces acting on the spacecraft [9]. They have been calculated as follows

$$F_{dR}(x_{EOE}, t) = m a_{dR}(x_{EOE}, t) \quad (54)$$

$$F_{dT}(x_{EOE}, t) = m a_{dT}(x_{EOE}, t) \quad (55)$$

$$F_{dN}(x_{EOE}, t) = m a_{dN}(x_{EOE}, t) \quad (56)$$

$$F_d(x_{EOE}, t) = \sqrt{F_{dR}(x_{EOE}, t)^2 + F_{dT}(x_{EOE}, t)^2 + F_{dN}(x_{EOE}, t)^2} \quad (57)$$

4. SIMULATION RESULTS

The Simulations are performed with a program implemented in Matlab. The corresponding results are presented in this work, to validate the models. The satellite considered in the simulations is characterized by the following structural parameters: a mass $m=4500$ kg; a mean surface absorbing the solar radiation $S=300$ m²; a mean reflectivity coefficient $\varepsilon = 0.3$, which entails a radiation pressure coefficient $C_R = 1.3$.

Fig. 2 to Fig. 7 show the equinoctial element time histories obtained by numerical integration of Gauss' VOP equations over 2 years with initial dynamical conditions $a(t_0)=a_k$, $P_1(t_0)=P_2(t_0)=Q_1(t_0)=Q_2(t_0)=0$, $l_\theta=60^\circ$.

At the initial epoch $t_0=0$ corresponding to the date 2010 January 1.0. for a satellite, a disturbing

acceleration vector and a t_θ value like the ones above.

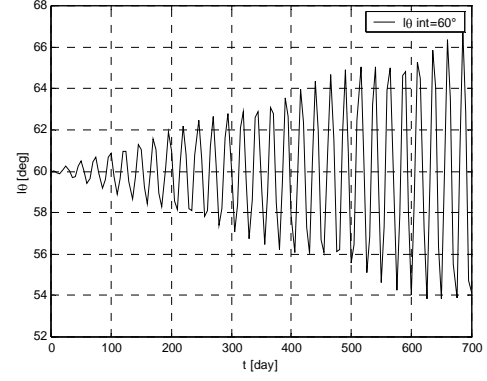


Fig. 2. Mean latitude l_θ time histories over 2 years.

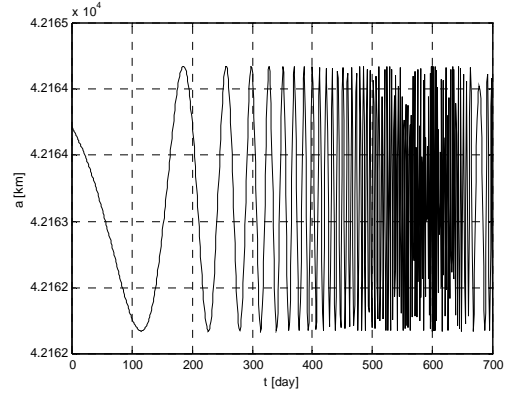


Fig. 3. Semi-major axis a time histories over 2 years.

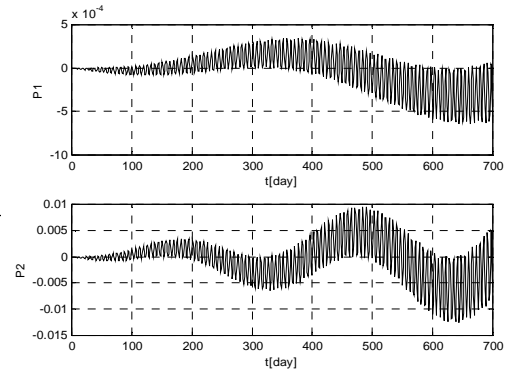


Fig. 4. Time histories of the eccentricity vector components P_1 and P_2 over 2 years.

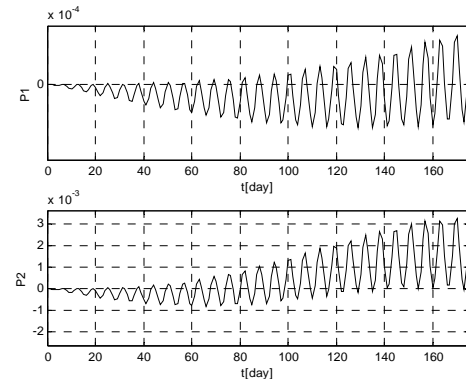


Fig. 5. Zoom of Fig. 4 over 6 weeks.

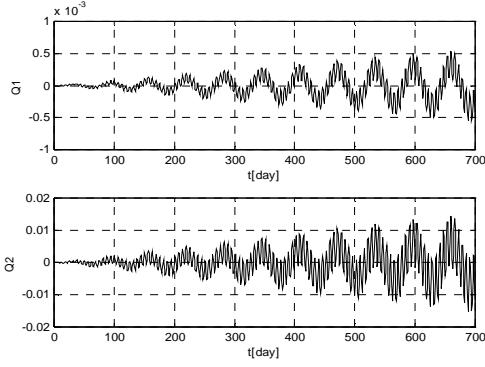


Fig. 6. Time histories of the inclination vector components Q_1 and Q_2 over 2 years.

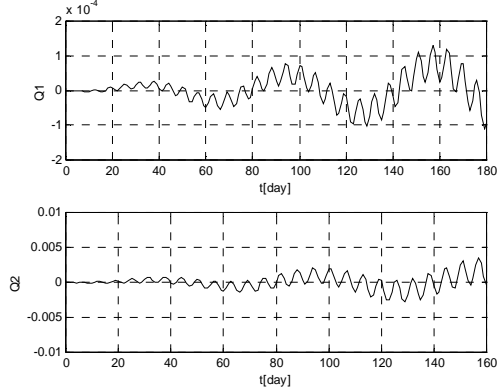


Fig. 7. Zoom of Fig. 6 over 6 weeks.

Fig. 8 and Fig. 9 show the time histories of F_{dR} , F_{dT} , F_{dN} and F_d over 2 years. We observe that the amplitudes of these force components vary in the same range for both the initial conditions and that they have the same periodic behavior.

Fig. 10 and Fig. 11 is a zoom of Fig. 8 and Fig. 9 over 6 weeks and they show that a change of initial longitude results in a change of phase of the force component time histories.

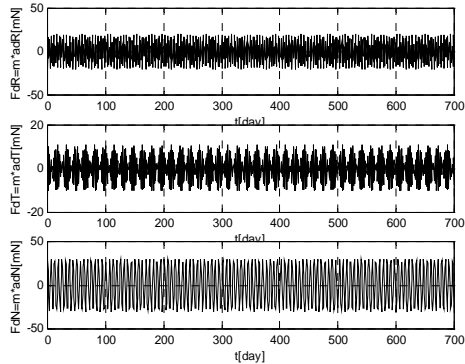


Fig. 8. Time history over 2 years of the Gaussian components of the environmental disturbing force vector acting on a spacecraft with mass $m = 4500$ kg.

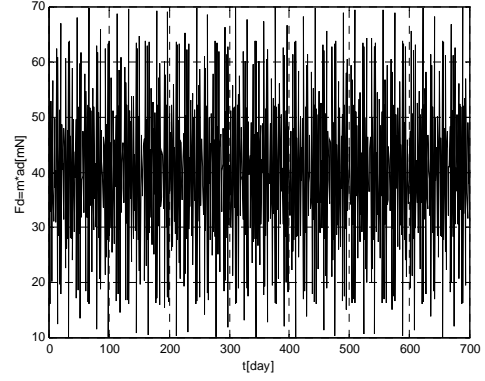


Fig. 9. Time history over 2 years of the modulus of the environmental disturbing force vector acting on a spacecraft with mass $m = 4500$ kg.

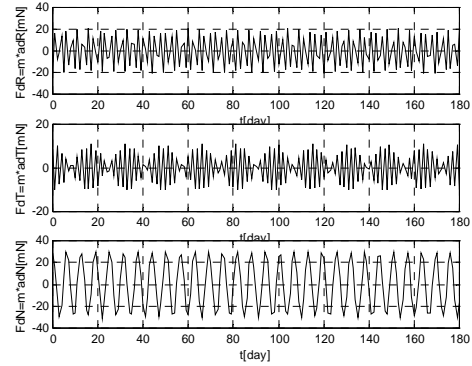


Fig. 10. Zoom of Fig. 7 over 6 weeks.

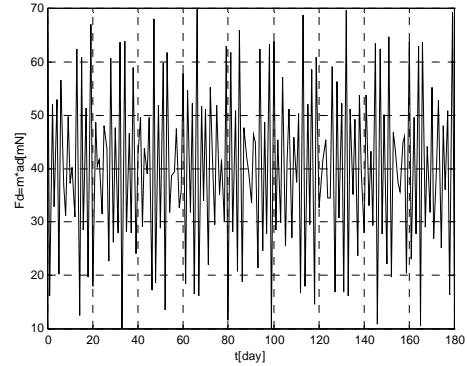


Fig. 11. Zoom of Fig. 8 over 6 weeks.

5. CONCLUSION

We have studied translation dynamical nonlinear models in unperturbed Keplerian conditions for a GEO satellite written in terms of environmental disturbing accelerations (Gauss' form).

Numerical simulations of these models let us obtain the time history of the satellite state vector and of the perturbing accelerations, which depend on orbital parameters; they allow us to identify also from which perturbing force the orbital parameters are mainly affected.

APPENDIX

$$a_k = 42164.172 \text{ km}$$

$$GM_{\oplus} = 398600.4415 \text{ km}^3/\text{s}^2$$

$$\omega_{\oplus} = 0.7292115 \times 10^{-4} \text{ rad/s}$$

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