ON PROPER SOLUTION OF THE TWO-SIDED MODEL MATCHING PROBLEM

K.H. Kiritsis
Hellenic Air Force Academy, Department of Aeronautical Sciences,
Division of Automatic Control, Airbase of Dekelia, ΤΓΑ 1010, Dekelia, Athens, Greece.
e-mail: kkyritsis.hafa@haf.gr   costas.kyritsis@gmail.com

Abstract: *In this paper the problem of two-sided model matching problem is studied. Necessary and sufficient conditions are given for the problem to have a solution over the Euclidean ring of proper rational functions. A simple procedure is given for the computation of the solution. Our approach is based on properties of rational matrices at infinity and it is computationally simple. Our results are useful for further understanding and the solution of unsolved fundamental problem of model matching by constant output feedback for linear time-invariant systems with controlled output different from measurement output and external disturbances. The main results of this paper and therefore model matching control by constant output feedback are useful in the design of modern aircraft control systems.*

Key words: *model matching, proper solution, infinite zero structure.*

1. Introduction

Let *A(z), B(z)* and *C(z)* be a given proper rational matrices of appropriate dimensions. The two-sided model matching problem is defined as follows. Does there exist a proper rational matrix **X**(z) such that

$$A(z)X(z)B(z) = C(z) \qquad (1)$$

If so, give necessary and sufficient conditions for existence and a procedure to calculate the proper rational matrix $X(z)$.

The two-sided model matching with stability was studied in [1] and [2] where necessary and sufficient conditions have been established for the existence of solution. In [3] under the assumption that the matrices *A(z)* and *B(z)* have full row rank and full column rank respectively, a sufficient condition has been established for the solution of two-sided model matching problem with stability and an efficient method for computation of solution has been given. The two-sided model problem has many applications in linear control theory [1], [2], [3], [4] and [5]. The purpose of this paper is to present a simple solution of the two-sided model matching problem over the Euclidean ring of proper rational functions. In particular, necessary and sufficient conditions are established which guarantee the existence of solution of (1) over the Euclidean ring of proper rational functions and an algorithm is given for the computation of solution.

Our approach is based on properties of proper rational matrices at infinity and has certain advances with respect to existing results in literature [1] and [2]. Firstly, it not only gives necessary and sufficient conditions for the existence of solution of two-sided model matching problem but also gives a procedure to find the solution over the Euclidean ring of proper rational functions. Secondly, it provides deeper insight into the problem, since is proved that the existence of solution of the two-sided model matching problem over the Euclidean ring of proper rational functions, depends on infinite zero structure of proper rational matrices *A(z), B(z)* and *C(z)*. In our point of view our results are useful for further understanding and the solution of unsolved fundamental problem of model matching by constant output feedback for linear time-invariant systems with controlled output different from measurement output and external disturbances [3]. The main results of this paper and therefore model matching control by constant output feedback are useful in the design of modern aircraft control systems [8]. The above clearly demonstrates the contribution of main results of this paper.

2. Basic concepts and preliminary results

Let $R_p(z)$, be the Euclidean ring of proper rational functions in *z*. A square matrix *U(z)* over $R_p(z)$, is said to be biproper if its inverse exists and is also proper. A matrix *W(z)* whose elements are proper rational functions is called proper rational matrix. A conceptual tool for the study of the structure of rational matrices is the following standard form. Every *p x m* proper rational matrix *W(z)* with rank[*W(z)*]=*r* can be expressed as

$$W(z) = U_1(z) \, M(z) \, U_2(z) \qquad (2)$$

where *U₁(z)* and *U₂(z)* are biproper matrices and the matrix *M(z)* is given by

$$M(z) = \begin{bmatrix} M_r(z) & 0 \\ 0 & 0 \end{bmatrix} \qquad (3)$$

and $M_r(z) diag\left[z^{-\delta 1}, \ldots\ldots, z^{-\delta r} \right]$. The numbers $\delta_1 \leq \ldots \leq \delta r$ are non-negative integers uniquely determined by *W(z)*. Relationship (2) is the Smith-McMillan form over $R_p(z)$, [6], [7] of proper rational matrix *W(z)* and the non-negative integers *δi* for *i = 1,2, ...r*, determine the structure of the infinite zero of the matrix *W(z)*.

The following Lemmas are taken from [6] and are

needed to prove the main theorem of this proper.

Lemma 1. Let $P(z)$ and $Q(z)$ be rational matrices with elements in $R_p(z)$, Then the equation

$$P(z)X(z) = Q(z) \qquad (4)$$

has a solution over $R_p(z)$ if and only if the matrices $P(z)$ and $[P(z), Q(z)]$ have the same infinite zero structure.

Lemma 2. Let $N(z)$ and $R(z)$ be rational matrices with elements in $R_p(z)$. Then the equation

$$Y(z)N(z) = R(z) \qquad (5)$$

has a solution over $R_p(z)$, if and only if the matrices $N(z)$ and $\begin{bmatrix} N(z) \\ R(z) \end{bmatrix}$ have the same infinite zero structure.

3. Main results

Let rank$[A(z)]=r$, then there exists biproper matrices $U_1(z)$ and $U_2(z)$ which reduce $A(z)$ to the Smith–McMillan form over $R_p(z)$

$$A(z) = U_1(z)\begin{bmatrix} A_r(z) & 0 \\ 0 & 0 \end{bmatrix} U_2(z) \qquad (6)$$

where $A_r(z) = diag\left[z^{-\delta 1}, \ldots \ldots, z^{-\delta r}\right]$.

The integers δi for $i = 1,2,\ldots, r$ determine the structure of infinite zero of $A(z)$. Denote

$$U_1^{-1}(z)\,C(z) = \begin{bmatrix} C_1(z) \\ C_2(z) \end{bmatrix} \qquad (7)$$

The following Theorem is the main result of this paper and provides necessary and sufficient conditions for the existence of a proper solution of two-sided model matching problem.

Theorem 1. The two-sided matching problem has a solution over $R_p(z)$ if and only if the following conditions hold:

(a) The matrices $A(z)$ and $[A(z), C(z)]$ have the same infinite zero structure.

(b) The matrices $B(z)$ and $\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix}$ have the same infinite zero structure.

Proof: Suppose that the two-sided model matching problem has a solution for $X(z)$ over $R_p(z)$. If we define $Y(z) = X(z)B(z)$ then equation (1) can be rewritten as follows

$$A(z)Y(z) = C(z) \qquad (8)$$

Since equation (1) has a solution for $X(z)$ over $R_p(z)$, then the equation (8) has also solution for $Y(z)$ over $R_p(z)$ and by Lemma 1 the matrices $A(z)$ and $[A(z), C(z)]$ have the some infinite zero structure. This is condition (a) of the Theorem.

Using (6) equation (1) can be rewritten as follows.

$$U_1(z)\begin{bmatrix} A_r(z) & 0 \\ 0 & 0 \end{bmatrix} U_2(z)\,X(z)\,B(z) = C(z) \qquad (9)$$

or equivalently

$$\begin{bmatrix} A_r(z) & 0 \\ 0 & 0 \end{bmatrix} U_2(z)\,X(z)\,B(z) = U_1^{-1}(z)\,C(z) \qquad (10)$$

Using (7) equation (10) can be rewritten as follows.

$$\begin{bmatrix} A_r(z) & 0 \\ 0 & 0 \end{bmatrix} U_2(z)\,X(z)\,B(z) = \begin{bmatrix} C_1(z) \\ C_2(z) \end{bmatrix} \qquad (11)$$

From (11) we have that

$$[A_r, 0]\,U_2(z)\,X(z)\,B(z) = C_1(z) \qquad (12)$$

or equivalently

$$\Phi(z)\,B(z) = A_r^{-1}(z)\,C_1(z) \qquad (13)$$

where the matrix $\Phi(z)$ over $R_p(z)$, is given by

$$\Phi(z) = [I_r \quad 0]\,U_2(z)\,X(z) \qquad (14)$$

Since $X(z)$ exists and is proper as well, equation (13) has a solution for $\Phi(z)$ over $R_p(z)$ and therefore according to Lemma 2 the matrices $B(z)$ *and* $\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix}$ have the same infinite zero structure. This is condition (b) of the Theorem.

The sufficiency of conditions (a) and (b) can be proved as follows. Condition (a) guarantees that the matrix $C_2(z)$ in (11) is zero. Since $C_2(z) = 0$ and the rational matrix $A_r(z)$ is nonsingular, equation (1) is equivalent to equation (13). Condition (b) guarantees that the equation (13) has a solution for $\Phi(z)$ over $R_p(z)$. If $\Phi(z)$ is a solution over $R_p(z)$ of (13), then from (14) it follows that the matrix

$$X(z) = U_2^{-1}(z)\begin{bmatrix} \Phi(z) \\ 0 \end{bmatrix} \qquad (15)$$

is a solution over $R_p(z)$) of equation (1). This completes the proof.

In what follows a procedure is described to determine a solution over $R_p(z)$ of the two-sided model matching problem.

4. Computational algorithm

Given: $A(z), B(z)$ and $C(z)$, find $X(z)$

Step 1. Let rank $[A(z)] = r$. Find biproper matrices $U_1(z)$ and $U_2(z)$ which reduce $A(z)$ to the Smith–McMillan form over $R_p(z)$.

$$A(z) = U_1(z)\begin{bmatrix} A_r(z) & 0 \\ 0 & 0 \end{bmatrix} U_2(z)$$

where $A_r(z) = diag\left[z^{-\delta 1}, \ldots \ldots, z^{-\delta r}\right]$. Denote

$$U_1^{-1}(z)\,C(z) = \begin{bmatrix} C_1(z) \\ C_2(z) \end{bmatrix}$$

where $C_1(z)$ has r rows.

Step 2. Check the conditions (a) and (b) of the

Theorem 1. If these conditions are satisfied go to step 3. If conditions of the Theorem 1 are not satisfied go to step 5.

Step 3. Solve equation (13) and find $\Phi(z)$

Step 4. Set

$$X(z) = U_2^{-1}(z)\begin{bmatrix} \Phi(z) \\ 0 \end{bmatrix}$$

Step 5. Our problem has no solution

5. A numerical example

To illustrate the computation of solution of two-sided model matching problem, consider equation (1) with matrices $A(z)$, $B(z)$ and $C(z)$ given by

$$A(z) = \begin{bmatrix} \frac{z-2}{z-1} & \frac{z-3}{z-2} \\ \frac{1}{z-1} & 0 \end{bmatrix}$$

$$B(z) = \begin{bmatrix} \frac{1}{z-2} & \frac{z}{z-2} \\ 0 & \frac{z+3}{z+1} \end{bmatrix}$$

$$C(z) = \begin{bmatrix} \frac{1}{z} & \frac{2z+4}{z+1} \\ \frac{1}{z(z-2)} & \frac{1}{(z-2)} \end{bmatrix}$$

Our aim is to find the solution for $X(z)$ of equation (1) over the Euclidean ring of proper rational functions.

We shall follow the Steps of computational algorithm given in the section 4. To execute Step 1, we compute the Smith–McMillan form over $R_p(z)$. of matrix $A(z)$. We have that

$$A(z) = \begin{bmatrix} \frac{z-2}{z-1} & \frac{z-3}{z-2} \\ \frac{1}{z-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{bmatrix}\begin{bmatrix} \frac{z-2}{z-1} & \frac{z-3}{z-2} \\ \frac{z}{z-1} & 0 \end{bmatrix} \quad (16)$$

We have that

$$A_r(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{bmatrix} \quad (17)$$

$$U_2(z) = \begin{bmatrix} \frac{z-2}{z-1} & \frac{z-3}{z-2} \\ \frac{z}{z-1} & 0 \end{bmatrix} \quad (18)$$

$$U_1(z) = I \quad (19)$$

where I is the identity matrix of dimensions 2 x 2. Since the matrix $C(z)$ is square and the matrix $U_1(z)$ is the identity matrix, from (7) we have that

$$C_1(z) = U_1^{-1}(z)\, C(z) = C(z) \quad (20)$$

To execute Step 2 we compute the Smith–McMillan form over $R_p(z)$.of matrix $[A(z),\ C(z)]$. We have

that

$$[A(z),C(z)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{z} & 0 & 0 \end{bmatrix}\begin{bmatrix} D(z) & K(z) \\ 0 & I \end{bmatrix}(21)$$

The matrix

$$\begin{bmatrix} D(z) & K(z) \\ 0 & I \end{bmatrix} \quad (22)$$

is biproper of dimensions 4 x 4. The matrix I and the zero matrix in (22) are square of dimensions 2 x 2. The matrices $D(z)$ and $K(z)$ in (22) of dimensions 2 x 2 are given by

$$D(z) = \begin{bmatrix} \frac{z-2}{z-1} & \frac{z-3}{z-2} \\ \frac{z}{z-1} & 0 \end{bmatrix} \quad (23)$$

$$K(z) = \begin{bmatrix} \frac{1}{z} & \frac{2z+4}{z+1} \\ \frac{1}{(z-2)} & \frac{z}{(z-2)} \end{bmatrix} \quad (24)$$

We form the matrix $A_r^{-1}(z)\, C_1(z)$. Using (20) and (17) we have that

$$A_r^{-1}(z)\, C_1(z) = A_r^{-1}(z)\, U_1^{-1}(z)\, C(z) =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} \frac{1}{z} & \frac{z+4}{z+1} \\ \frac{1}{z(z-2)} & \frac{1}{(z-2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{z} & \frac{2z+4}{z+1} \\ \frac{1}{(z-2)} & \frac{z}{(z-2)} \end{bmatrix}(25)$$

We compute the Smith–McMillan form over $R_p(z)$. of matrix $B(z)$. We have that

$$B(z) = \begin{bmatrix} \frac{1}{z-2} & \frac{z}{z-2} \\ 0 & \frac{z+3}{z+1} \end{bmatrix} = \begin{bmatrix} \frac{z}{z-2} & \frac{z}{z-2} \\ 0 & \frac{z+3}{z+1} \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{bmatrix} \quad (26)$$

We compute the Smith–McMillan form of matrix $\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix}$ over $R_p(z)$. We have that

$$\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix} = \begin{bmatrix} E(z) & 0 \\ F(z) & I \end{bmatrix}\begin{bmatrix} \frac{1}{z} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (27)$$

where the matrix

$$\begin{bmatrix} E(z) & 0 \\ F(z) & I \end{bmatrix} \quad (28)$$

is biproper of dimensions 4 x 4. The matrix I and the zero matrix in (28) are square of dimensions 2 x 2. The matrices $E(z)$ and $F(z)$ in (28) of dimensions 2 x 2 are given by

$$E(z)= \begin{bmatrix} \frac{z}{z-2} & \frac{z}{z-2} \\ 0 & \frac{z+3}{z+1} \end{bmatrix} \qquad (29)$$

$$F(z) = \begin{bmatrix} 1 & \frac{2z+4}{z+1} \\ \frac{z}{(z-2)} & \frac{z}{(z-2)} \end{bmatrix} \qquad (30)$$

From (6) and (16) we have that the infinite zero structure of $A(z)$ is $\{0,1\}$ and from (6) and (21) we have that the infinite zero structure $[A(z), C(z)]$ is $\{0,1\}$. From the above it follows that $A(z)$ and $[A(z), C(z)]$ have the same infinite zero structure and therefore condition (a) of Theorem is verified.

From (6) and (26) we have that the infinite zero structure of $B(z)$ is $\{1,0\}$ and from (6) and (27) we have that the infinite zero structure $\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix}$ is $\{1,0\}$. From the above it follows that $B(z)$ and $\begin{bmatrix} B(z) \\ A_r^{-1}(z)C_1(z) \end{bmatrix}$ have the same infinite zero structure and therefore condition (b) of Theorem is verified. Hence the two-sided matching problem has a solution over the Euclidean ring of proper rational functions.

To execute Step 3, we form the matrix $\Phi(z)$ given by (14). Since rank $[A(z)] = 2$. We have that

$$\Phi(z) = I_2 \ U_2(z) \ X(z) = U_2(z) \ X(z) \qquad (31)$$

Since the matrix $B(z)$ is nonsingular, equation (13) has a unique solution for $\Phi(z)$ given by

$$\Phi(z) = A_r^{-1}(z) \ C_1(z) B^{-1}(z) \qquad (32)$$

Using (20) relationship (32) can be rewritten as follows

$$\Phi(z) = A_r^{-1}(z) \ C(z) B^{-1}(z) \qquad (33)$$

To execute Step 4, we compute matrix $X(z)$ from (31) as follows

$$X(z) = U_2^{-1}(z) \ \Phi(z) \qquad (34)$$

From (34) using (33) we have that

$$X(z) = U_2^{-1}(z) \ A_r^{-1}(z) \ C(z) B^{-1}(z) =$$

$$= \begin{bmatrix} \frac{z-1}{z} & 0 \\ 0 & \frac{z-2}{z-3} \end{bmatrix} \qquad (35)$$

From (35) we have that $X(z)$ is a matrix over $R_p(z)$ and the solution of numerical example is complete.

6. Conclusions

In this paper, the two-sided model matching problem is studied and completely solved. In particular, necessary and sufficient conditions have been established for the problem to have a proper solution over the Euclidean ring of proper rational functions and a simple procedure is given for the computation of the solution. Our results are useful for further understanding of two-sided model matching problem and have certain advances with respect to existing results regarding the solvability of this fundamental problem in linear control theory.

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